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Dedication

I dedicate this dissertation to my family.

My beloved parents and my caring wife, for their unlimited support, encouregement and patience.

To my supportive brothers, friends and classmates.

A special thanks to all of my teachers and professors who taught me and guided me from the beginning of my academic journey until this graduation.

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Contents

Introduction

The study of dynamic equations is a wide field in pure and applied mathematics. All of these disciplines are concerned with the properties of these equations of various types. Pur mathematics focuses on the existence and uniqueness of solution. Applied mathematics emphasizes the rigorous justification of the qualitative behavior of solutions (oscillation, periodic orbits, stability, etc). The oscillation theory as a part of the qualitative theory of dynamic equations.

The theory of oscillation of the solutions of neutral differential and difference equations presents a strong theoretical interest. One reason for this is that they arise in several areas of applied mathematics including circuit theory, phenomena in technology, natural and social sciences. Additionally, other real fields where neutral diff erential equations, as for instance, are invoked in theoretical physics, population dynamics, biomathematics, chemistry, and engineering. Moreover, moment problem approaches appear also as a natural instrument in control theory of neutral type systems; see [68, 69, 70] and [71], respectively).

A time scale is any nonempty closed subset of the real line. The theory of time scales is a fairly new area of research. It was introduced in Stefan Hilger's 1988 Ph.D. thesis [62], as a way to unify the seemingly disparate fields of discrete dynamical systems (i.e., difference equations) and continuous dynamical systems (i.e., differential equations).Today it is better known as the time scale calculus. Since the nineties of XX century, the study of dynamic equations on time scales received a lot of attention. In 1997, the German mathematician Martin Bohner came across time scale calculus by chance, when he took up a position at the National University of Singapore. On the way from Singapore airport, a colleague, Ravi Agarwal, mentioned that time scale calculus might be the key to the problems that Bohner was investigating at that time. After that episode, time scale calculus became one of its main areas of research. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [49, 50], summarize and organize much of time scale calculus.

In this thesis, we present oscillation of different types of first-order linear dynamic equations on time scales and fourth-order and hybrid nonlinear functional dynamic equations with damping.

We have organized this thesis as follows:

In Chapter 1, we present some definitions and results which are used throughout this thesis.

In Chapter 2: Oscillation theorems for fourth-order hybrid nonlinear functional dynamic equations with damping. In this chapter, we will establish some oscillation criteria for the fourth order hybrid nonlinear functional dynamic equations with damping

$$\left(\frac{a\left(t\right)\left(u^{(2)}\left(t\right)\right)^{\beta}}{f\left(t,u\left(t\right)\right)}\right)^{(2)}+\sum\nolimits_{i=1}^{i=n}b_i\left(t\right)u^{\beta}\left(\tau_i\left(t\right)\right)=g\left(t,u\left(\eta\left(t\right)\right)\right),\quad \text{for all } t\in[t_0,\infty), \tag{1}$$

where n is an integer and β is a quotient of odd integer, such as $\beta>0$ and $n\geq 1$.

We establish new oscillation criteria to check whether all solutions of an equation, in this class, oscillate. This study aims to present some new sufficient conditions for the oscillatory of solutions to a class of fourth-order hybrid nonlinear functional dynamic equations by use of Riccati technique and other method.

In Section 2.2, we establish new oscillation results for Equation (2.1) while in final section. Some examples and discussions are discussed in Section 2.3.

In Chapter 3: Oscillation theorems for advanced differential equations. In this chapter, we will establish some oscillation criteria for the advanced differential equations

$$u^{'}\left(t\right)-\sum\nolimits_{i=1}^{i=k}q_i\left(t\right)u^{\alpha}\left(\tau_i\left(t\right)\right)=0,\quad \text{for } t\in[t_0,\infty) \tag{2}$$

where k is an integer and α is a quotient of odd integer, such that $k\geq 1$ and $\alpha\geq 1$. The functions $\{q_i\}$, for $i\in\{1,...,k\}$ are continuous positive functions and the arguments $\{\tau_i\}$, for $i\in\{1,...,k\}$ are continuous positive functions, such that $\tau_i\left(t\right)>t$, for for $i\in\{1,...,k\}$. This study aims to present some new sufficient conditions for the oscillation of solutions to a class of first-order advanced differential equations, using a technique based on a recursive sequence.

In Chapter 4: An improved oscillation result for advanced differential equations on time scale. In this chapter, we will establish some oscillation criteria for for advanced differential equations on time scale

$$u^{\Delta}(t) - \eta(t) u(\lambda(t)) = 0, \quad \text{for } t \in [t_0, \infty) \cap \mathbb{T}, \tag{3}$$

on a time scale $\mathbb{T}$, where $\sup \mathbb{T} = \infty$, $t_0 \in \mathbb{T}$. The functions $\eta \in \mathcal{C}([t_0, \infty) \cap \mathbb{T}, [0, \infty))$ and $\lambda \in \mathcal{C}([t_0, \infty) \cap \mathbb{T}, [t_0, \infty) \cap \mathbb{T})$, such as $\eta \neq 0$ on any interval of the form $[t_0, \infty) \cap \mathbb{T}$, $\lambda(t) > t$, for $t \in [t_0, \infty) \cap \mathbb{T}$ and $\lim_{t \to \infty} \lambda(t) = \infty$.

We present some new sufficient conditions for the oscillatory of solutions to a class of first-order advanced differential equation on time scale.

Publications

[1] A. Benaissa Cherif, **M. Fethallah**, and F. Z. Ladrani, Oscillation theorems for fourth-order hybrid nonlinear functional dynamic equations with Damping, Bulletin of the Transilvania University of Brasov, Series III: Mathematics and Computer Science, Vol. 1(63), No. 1 - 2021, 53-70.

[2] A. Benaissa Cherif, **M. Fethallah**, and F. Z. Ladrani, Oscillation theorems for advanced differential equations, Novi Sad J. Math. Vol. XX, No. Y, 20ZZ.

[3] **M. Fethallah**, A. Benaissa Cherif, An improved oscillation result for advanced differential equations on time scale, new Trends in Mathematical Sciences, NTMSCI 10 Special Issue, No. 1, 6-12 (2022).

Chapter 1

Preliminaries

In this chapter, we present some definitions and results which we will use in this Thesis.

1.1 Time Scales

In this Section , we introduce the calculus on time scales and we also present the differentiability, integration and exponential function on time scales. The reader interested on the subject is referred to the books [49, 50].

1.1.1 Basic definitions

Definition 1.1.1. *[49, 50]A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.*

The following operators of time scales theory are used, in literature and throughout this thesis, several times:

Definition 1.1.2. *[49, Definition 1.1]The mapping $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$, if $\mathbb{T}$ has a maximum t) is called the forward jump operator.*
Accordingly, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ with $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). The symbol $\emptyset$ denotes the empty set.

The following classification of points is used within the theory: a point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$,

$\rho(t) < t$, respectively. A point t is called isolated if $\rho(t) < t < \sigma(t)$ and dense if $\rho(t) = t = \sigma(t)$.

Definition 1.1.3. *[49, 50]The forward graininess function $\mu : \mathbb{T} \to [0, +\infty[$ is defined by*

$$\mu(t) = \sigma(t) - t.$$

Example 1. *If $\mathbb{T} = \mathbb{R}$, then $\rho(t) = \sigma(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = h\mathbb{Z}$, then $\rho(t) = t - h$, $\sigma(t) = t + h$ and $\mu(t) = h, \quad h > 0$.*

Now, let us define the sets $\mathbb{T}^{k^n}$, inductively:

$$\mathbb{T}^{k^1} = \mathbb{T}^k = \{t \in \mathbb{T} : t \text{ non-maximal or left-dense}\}$$

and $\mathbb{T}^{k^n} = \left(\mathbb{T}^{k^{n-1}}\right)^k$, $n \geq 2$.

1.1.2 Differentiation

Throughout the thesis we will frequently write $f^\sigma(t) = f(\sigma(t))$. Next results are related with differentiation on time scales.

Definition 1.1.4. *[49, Definition 1.10]Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\triangle(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $\mathcal{U}$ of t (i.e., $\mathcal{U} = (t - \delta, t + \delta) \cap \mathbb{T}$, for some $\delta > 0$) such that*

$$\left|[f(\sigma(t)) - f(s)] - f^\triangle(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|, \qquad \text{for all } s \in \mathcal{U}.$$

We call $f^\triangle(t)$ the delta derivative of f at t.
Moreover, we say that f is delta differentiable on $\mathbb{T}^k$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^\triangle : \mathbb{T}^k \longrightarrow \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^k$.

Some basic properties will now be given for the $\triangle$-derivative.

Theorem 1.1.1. *[49, Theorem 1.6]Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\triangle(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(*iii*) *If* t *is right-dense, then* f *is differentiable at* t *if the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case

$$f^{\triangle}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(*iv*) *If* f *is differentiable at* t*, then*

$$f(\sigma(t))=f(t)+\mu(t)f^{\triangle}(t).$$

It is an immediate consequence of Theorem 1.1.1 that if $\mathbb{T}=\mathbb{R}$, then the $\triangle$-derivative becomes the classical one, i.e., $f^{\triangle}=f'$, while if $\mathbb{T}=\mathbb{Z}$, the $\triangle$-derivative reduces to the forward difference $f^{\triangle}(t)=\triangle f(t)=f(t+1)-f(t)$.

Theorem 1.1.2. *[49, Theorem 1.20]Assume* $f,g:\mathbb{T}\to\mathbb{R}$ *are differentiable at* $t\in\mathbb{T}^k$*, then*

(*i*) *Then sum* $f+g:\mathbb{T}\to\mathbb{R}$ *is differentiable at* t *with*

$$(f+g)^{\triangle}(t)=f^{\triangle}(t)+g^{\triangle}(t).$$

(*ii*) *For any constant* α*,* $\alpha f:\mathbb{T}\to\mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^{\triangle}=\alpha f^{\triangle}(t).$$

(*iii*) *The product* $fg:\mathbb{T}\to\mathbb{R}$ *is differentiable at* t *with*

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t)+f^{\sigma}(t)g^{\triangle}(t)\\ &= f^{\triangle}(t)g^{\sigma}(t)+f(t)g^{\triangle}(t).\end{aligned}$$

(*iv*) *If* $f(t)f(\sigma(t))\neq 0$*, then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\triangle}(t)=-\frac{f^{\triangle}(t)}{f(t)f(\sigma(t))}.$$

(*v*) *If* $g(t)g(\sigma(t))\neq 0$*, then* $\frac{1}{g}$ *is differentiable at* t *and*

$$\left(\frac{f}{g}\right)^{\triangle}(t)=\frac{f^{\triangle}(t)g(t)-f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}.$$

Theorem 1.1.3. *[49, Theorem 1.24]Let α be constant and $m \in \mathbb{N}$. Then we have the following:*

(i) For f defined by $f(t) = (t-\alpha)^m$ we have

$$f^{\triangle}(t) = \sum_{v=0}^{v=m-1} (\sigma(t)-\alpha)^v (t-\alpha)^{m-1-v}.$$

(ii) For g defined by $g(t) = \dfrac{1}{(t-\alpha)^m}$ we have

$$g^{\triangle}(t) = -\sum_{v=0}^{v=m-1} \frac{1}{(\sigma(t)-\alpha)^{m-v}(t-\alpha)^{v+1}},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

Definition 1.1.5. *[49, Definition 1.27]For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\triangle\triangle}$ provided $f^{\triangle}$ is differentiable on $\mathbb{T}^{k^2}$, with derivative $f^{\triangle\triangle} = \left(f^{\triangle}\right)^{\triangle} : \mathbb{T}^{k^2} \longrightarrow \mathbb{R}$. Similarly we define higher order derivatives $f^{\triangle^n} : \mathbb{T}^{k^n} \longrightarrow \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote*

$$\sigma^n(t) = \sigma^{n-1} \circ \sigma, \quad \text{and} \quad \rho^n(t) = \rho^{n-1} \circ \rho, \quad \text{for } n \in \mathbb{N}.$$

For convenience we also put,

$$\sigma^0(t) = t, \quad \rho^0(t) = t, \quad f^{\triangle^0} = f, \quad \text{and} \quad \mathbb{T}^{k^0} = \mathbb{T}.$$

Theorem 1.1.4 (Leibniz Formula). *[49, Theorem 1.32]Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. If f^{Λ} exists for all $\Lambda \in S_k^{(n)}$, then*

$$(fg)^{\triangle^n} = \sum_{k=0}^{k=n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\triangle^k}, \text{ for all } n \in \mathbb{N}$$

Now we present a chain rule which calculates $(f \circ g)^{\triangle}$, where

$$g : \mathbb{T} \to \mathbb{R} \qquad \text{and} \qquad f : \mathbb{R} \to \mathbb{R}.$$

This chain rule is due to Christian Pötzsche, who derived it first in 1988.

Theorem 1.1.5. *[49, Theorem 1.90]Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. Then $f \circ g$ is Δ-differentiable and the formula*

$$(f \circ g)^{\triangle}(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t) g^{\triangle}(t)\right) dh \right\} g^{\triangle}(t)$$

holds.

1.1.3 Integration

Definition 1.1.6. *[49, Definition 1.57]A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Example 2. *Let $\mathbb{T} = \{0, 2\} \cup \left\{ \frac{1}{n}, 2 - \frac{1}{n} : n \in \mathbb{N}^* \right\}$ and define $f : \mathbb{T} \to \mathbb{R}$ by*

$$f(t) = \begin{cases} 0, & t \neq 2, \\ t, & t = 2. \end{cases}$$

Function f is regulated.

Definition 1.1.7. *[49, Definition 1.58]A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by*

$$\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of function $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathcal{C}^1_{rd} = \mathcal{C}^1_{rd}(\mathbb{T}) = \mathcal{C}^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 1.1.6. *[49, Theorem 1.60]Assume $f : \mathbb{T} \to \mathbb{R}$. Then we have the following:*

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If is regulated or rd-continuous, then so is f^σ.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Theorem 1.1.7 (Existence of Pre-Antiderivatives)**.** *[49, Theorem 1.60]Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation $\mathcal{D}$ such that*

$$F^{\triangle}(t) = f(t), \qquad \text{for all} t \in \mathcal{D}.$$

Definition 1.1.8. *[49, Definition 1.71]Assume $f:\mathbb{T}\to\mathbb{R}$ is a regulated function. Any function F is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by :*

$$\int f(t)\,\triangle t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.
We define the Cauchy integral by

$$\int_r^s f(t)\,\triangle t = F(s) - F(r)\,, \qquad \text{for all}\, r, s \in \mathbb{T}.$$

A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided

$$F^{\triangle}(t) = f(t)\,, \qquad \text{for all}\, t \in \mathbb{T}^k.$$

Theorem 1.1.8 (Existence of Antiderivative)**.** *[49, Theorem 1.74]Every rd-continuous function has an antiderivative. In particular if $t_0\in\mathbb{T}$ then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\,\triangle\tau, \qquad \text{for all}\, t\in\mathbb{T}$$

is an antiderivative of f.

Remark. *If $f\in\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$ and $t\in\mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\,\triangle\tau = \mu(t)\,f(t)\,.$$

Theorem 1.1.9. *[49, Theorem 1.77]If $a,b,c\in\mathbb{T}$, and $f,g\in\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$, then we have the following*

(i) $\displaystyle\int_a^b [\alpha f(t) + g(t)]\,\triangle t = \int_a^b f(t)\,\triangle t + \int_a^b g(t)\,\triangle t,$

(ii) $\displaystyle\int_a^b f(t)\,\triangle t = \int_a^c f(t)\,\triangle t + \int_c^b f(t)\,\triangle t,$

(iii) $\displaystyle\int_a^b f(\sigma(t))\,g^{\triangle}(t)\,\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)\,g(t)\,\triangle t.$

Theorem 1.1.10. *[49, Theorem 1.79]Let $a,b\in\mathbb{T}$ and $f\in\mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$.*

(i) *If $\mathbb{T}=\mathbb{R}$, then*

$$\int_a^b f(t)\,\triangle t = \int_a^b f(t)\,dt,$$

where the integral on he right is the usual Riemann integral from calculus.

(ii) If $[a,b]_{\mathbb{T}}$ consists of only isolated points, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{t\in[a,b)} \mu(t) f(t) & if a < b \\ 0 & if a = b \\ -\sum_{t\in[a,b)} \mu(t) f(t) & if a > b. \end{cases}$$

Example 3. *If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h>0$, then*

$$\int_a^b f(t)\triangle t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh) & if a < b \\ 0 & if\ a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh) & if a > b. \end{cases}$$

Definition 1.1.9. *[49, Definition 1.82]If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a,\infty)_{\mathbb{T}}$, then we define the improper integral by*

$$\int_a^\infty f(t)\triangle t := \lim_{x\to\infty} \int_a^x f(t)\triangle t$$

provided this limit exists, and we say that improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

1.1.4 The Exponential Function

Definition 1.1.10. *[49, Definition 2.25]We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided*

$$1 + \mu(t) p(t) \neq 0, \qquad for\, all\, t \in \mathbb{T}^k.$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this book by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T},\mathbb{R}).$$

Definition 1.1.11. *[49, Definition 2.30]If $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$, then we define the exponential function by*

$$e_p(t,s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\triangle\tau\right), \qquad for\, all\, s,t \in \mathbb{T},$$

where the cylinder transformation ξ_h is introduced in as in

$$\xi_h(z) = \begin{cases} \dfrac{1}{h}\log(1+zh), & if\, h > 0 \\ z & if\, h = 0 \end{cases}$$

Theorem 1.1.11. *[49, Theorem 2.33]Suppose $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$ and fix $t_0 \in \mathbb{T}$. Then $e_p(.,t_0)$ is a solution of the initial value problem*

$$\begin{cases} y^{\triangle} = p(t)\,y, \\ y(t_0) = 1 \end{cases} \quad on\ \mathbb{T}.$$

Theorem 1.1.12. *[49, Theorem 2.36]If $p, q \in \mathcal{R}(\mathbb{T},\mathbb{R})$ and $a, b, c \in \mathbb{T}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)\,p(t))\,e_p(t,s)$,

(iii) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$, *with* $\ominus p(t) = \dfrac{-p(t)}{1+\mu(t)\,p}$,

(iv) $e_p(t,s)\,e_p(s,r) = e_p(t,r)$,

(v) $e_p(t,s)\,e_q(t,s) = e_{p\oplus q}(t,s)$, *with* $p \oplus q = p(t) + q(t) + \mu(t)\,p(t)\,q(t)$,

(vi) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$, *with* $p \ominus q = p \oplus (\ominus q)$,

(vii) $\left(\dfrac{1}{e_p(.,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(.,s)}$.

Theorem 1.1.13. *[49, Theorem 2.44]Assume $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$ and $t_0 \in \mathbb{T}$, then we have the following:*

(i) If $1 + \mu p > 0$ on $\mathbb{T}^k$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.

(ii) If $1 + \mu p < 0$ on $\mathbb{T}^k$, then $e_p(t,t_0) = \alpha(t,t_0)(-1)^{n_t}$ for all $t \in \mathbb{T}$, where

$$\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \frac{1}{\mu(\tau)} \ln|1 + \mu(\tau)\,p(\tau)|\,\triangle\tau\right) > 0$$

and

$$n_t = \begin{cases} |[t_0,t)| & if\, t \geq t_0 \\ |[t,t_0)| & if\, t < t_0. \end{cases}$$

Definition 1.1.12. *[49, Definition 2.45]We define the set $\mathcal{R}^+(\mathbb{T},\mathbb{R})$ of all positively regressive elements of $\mathcal{R}(\mathbb{T},\mathbb{R})$ by*

$$\mathcal{R}^+(\mathbb{T},\mathbb{R}) := \{p \in \mathcal{R}(\mathbb{T},\mathbb{R}) : 1 + \mu(t)\,p(t) > 0,\ for\, all\, t \in \mathbb{T}\}.$$

Theorem 1.1.14 (Sign of Exponential Function)**.** *[49, Theorem 2.48]Let $p \in \mathcal{R}(\mathbb{T},\mathbb{R})$ and $t_0 \in \mathbb{T}$.*

(i) If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})^{+}$*, then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$.

(ii) If $1 + \mu(t) p(t) < 0$ *for some* $t \in \mathbb{T}^k$*, then* $e_p(t, t_0) e_p(\sigma(t), t_0) < 0$.

(iii) If $1 + \mu(t) p(t) < 0$ *for all* $t \in \mathbb{T}^k$*, then* $e_p(t, t_0)$ *changes sign at every points* $t \in \mathbb{T}$.

For more on the calculus on time scales, we refer the reader to the books [49, 50].

1.2 Elements of Analysis

For simplification, we note

$$\mathcal{D} := \{(t, s) \in \mathbb{R} : t \geq s \geq t_0\}$$

$$\mathcal{D}_0 := \{(t, s) \in \mathbb{R} : t > s \geq t_0\}.$$

Definition 1.2.1. *[11]The function* $H \in \mathcal{C}(\mathcal{D}, \mathbb{R})$ *is said to belongs to the function class* P *if*

i) $H(t, t) = 0$*, for* $t \in [t_0, \infty)$ *and* $H(, s) > 0$ *on* $\mathcal{D}_0$*,*

ii) H *has a nonpositive continuous and partial derivative* $\dfrac{\partial H}{\partial s}(t, s)$ *on* $\mathcal{D}_0$ *and there exist function* $\eta \in C^1([t_0, \infty), \mathbb{R})$*, such that*

$$\frac{\partial H(t, s)}{\partial s} + H(t, s)\frac{\eta'(s)}{\eta(s)} = \frac{h(t, s)}{\eta(s)} H(t, s). \tag{1.1}$$

Lemma 1.2.1 (Kiguarde's Lemma)**.** *[65, Theorem 2.2]. Let* $n \in \mathbb{N}$ *and* $f \in \mathcal{C}^n([t_0, \infty), \mathbb{R})$*. Suppose that* f *is either positive or negative and* $f^{(n)}$*is not identically zero and is either nonnegative or nonpositive on* $[t_0, \infty)$*. Then there exist* $t_1 \in [t_0, \infty)$*,* $m \in \{0, ..., n-1\}$ *such that* $(-1)^{n-m} f(t) f^{(n)}(t) \geq 0$ *holds for all* $t \in [t_1, \infty)$ *with*

(1) $f(t) f^{(j)}(t) \geq 0$ *holds for all* $t \in [t_1, \infty)$ *and* $j \in \{0, ..., m-1\}$*,*

(2) $(-1)^{m+j} f(t) f^{(j)}(t) \geq 0$ *holds for all* $t \in [t_1, \infty)$ *and* $j \in \{m, ..., n-1\}$.

Lemma 1.2.2. *[65, Lemma 2.3] Let* $f \in \mathcal{C}^n(\mathbb{T}, \mathbb{R})$*, with* $n \geq 2$*. Moreover, suppose that Kiguarde's Lemma 1.2.1 holds with* $m \in \{1, ..., n-1\}$ *and* $f^{(n)} \leq 0$ *on* $[t_0, \infty)$*. Then there exists a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$f^{(1)}(t) \geq \frac{(t - t_1)^{m-1}}{(m-1)!} f^{(m)}(t), \qquad \textit{for all } t \in [t_1, \infty).$$

Corollary 1.2.1. *[65, Corollary 2.4]Assume that the conditions of Lemma 1.2.2 hold. Then*

$$f(t) \geq \frac{(t-t_1)^m}{m!} f^{(m)}(t), \qquad \text{for all } t \in [t_1, \infty)_{\mathbb{T}}.$$

Lemma 1.2.3. *[9]If $n \in \mathbb{N}$ and $f \in C^n([t_0, \infty), \mathbb{R})$ then the following statements are true.*

(1) $\liminf\limits_{t\to\infty} f^{(n)}(t) > 0$ *implies* $\lim\limits_{t\to\infty} f^{(k)}(t) = \infty$, *for all* $k \in \{1, ..., n-1\}$.

(2) $\limsup\limits_{t\to\infty} f^{(n)}(t) < 0$ *implies* $\lim\limits_{t\to\infty} f^{(k)}(t) = -\infty$, *for all* $k \in \{1, ..., n-1\}$.

Lemma 1.2.4. *[26, Lemma 1]Consider the increasing sequence $(A_n)_n$ of positive real numbers, defined as follows*

$$A_{n+1} = a \exp(A_n), \qquad \text{with} \qquad A_0 = a > 0$$

Then $(A_n)_n$ converges in $\mathbb{R}^+$ if and only if $a \leq \frac{1}{e}$.

Next, we need the following lemma see [36].

Lemma 1.2.5. *[36] If A and B are nonnegative and $\lambda > 0$, then*

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda - 1) B^{\lambda}, \tag{1.2}$$

where equality holds if and if $A = B$.

Chapter 2

Oscillation theorems for fourth-order hybrid nonlinear functional dynamic equations with damping

In this chapter, we are dealing with the oscillation of the solutions of the fourth-order hybrid nonlinear functional dynamic equations with damping (2.1) by using the generalized Riccati transformations and an integral averaging method, the contribution is orginnal, as no results on the oscillation of fourth-order hybrid nonlinear functional dynamic equations having been reported in the literature.

2.1 Introduction

In this chapter, we extend the results of [3] to the oscillation of solutions of the fourth-order hybrid nonlinear dynamic equation

$$\left(\frac{a\left(t\right)\left(u^{(2)}\left(t\right)\right)^{\beta}}{f\left(t,u\left(t\right)\right)}\right)^{(2)}+\sum_{i=1}^{i=n}b_{i}\left(t\right)u^{\beta}\left(\tau_{i}\left(t\right)\right)=g\left(t,u\left(\eta\left(t\right)\right)\right),\quad \text{for all } t\in[t_{0},\infty), \tag{2.1}$$

where n is an integer and β is a quotient of odd integer, such as $\beta > 0$ and $n \geq 1$. Since we are interested in oscillation, we assume throughout this chapter that the given interval of the form $[t_0, \infty)$.

The equation (2.1) will be studied under the following assumptions:

(C_1) The function $f : [t_0, \infty) \times \mathbb{R}- \{0\} \to R$ such that $f \in \mathcal{C}([t_0, \infty) \times \mathbb{R}- \{0\}, \mathbb{R})$, $uf(t, u) > 0$, for all $(t, u) \in [t_0, \infty) \times \mathbb{R}- \{0\}$ and there is $\rho \in \mathcal{C}([t_0, \infty), [0, \infty))$ such that

$$f(t, u) \geq \rho(t), \quad \text{for all } (t, u) \in [t_0, \infty) \times \mathbb{R}- \{0\}. \tag{2.2}$$

(C_2) The function $g : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that $g \in \mathcal{C}([t_0, \infty) \times \mathbb{R}, \mathbb{R})$, $ug(t, u) > 0$, for all $(t, u) \in [t_0, \infty) \times \mathbb{R}- \{0\}$ and there is $\sigma \in \mathcal{C}([t_0, \infty), [0, \infty))$ such that

$$u^{-\beta} g(t, u) \leq \sigma(t), \quad \text{for all } (t, u) \in [t_0, \infty) \times \mathbb{R}- \{0\}. \tag{2.3}$$

(C_3) $a, \{b_i\}_{i \in \{1,..,n\}} \in \mathcal{C}([t_0, \infty), [0, \infty))$, such as

$$\int_{t_0}^{\infty} \left(\frac{s\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds = \infty. \tag{2.4}$$

and

$$B_n(t) := \sum_{i=1}^{i=n} b_i(t) - \sigma(t) \geq 0, \quad \text{for all } t \in [t_0, \infty). \tag{2.5}$$

(C_4) $\{\tau_i\}_{i \in \{1,..,n\}}, \eta \in \mathcal{C}([t_0, \infty), [t_0, \infty))$ such as τ, η are strictly increasing,

$$\lim_{t \to \infty} \tau_i(t) = \lim_{t \to \infty} \eta(t) = \infty.$$

and

$$\eta(t) \leq t \leq \tau_i(t), \quad \text{for all } t \in [t_0, \infty), \quad \text{for all } i \in \{1, ..., n\} \tag{2.6}$$

By a solution of (2.1) we mean a nontrivial real-valued function $u \in \mathcal{C}^4([T_u, \infty), \mathbb{R})$, $T_u \in [t_0, \infty)$ which satisfies (2.1) on $[T_u, \infty)$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (2.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (2.1) is called oscillatory if all its solutions are oscillatory.

On the other hand, the types of equations considered in the relevant literature are generally as follows. Using a comparison technique, J. Džurina et al. [42] studied the oscillation of solutions to fourth-order trinomial delay differential equations

$$y^{(4)}(t) + p(t) y^{'}(t) + q(t) y(\tau(t)) = 0, \quad \text{for all } t \in [t_0, \infty),$$

A. B. Trajkovict al. [1] studied the oscillatory behavior of intermediate solutions of Fourth-order nonlinear differential equations

$$\left(p(t) \left| x^{(2)}(t) \right|^{\alpha-1} x^{(2)}(t) \right)^{(2)} + q(t) \left| x(t) \right|^{\beta-1} x(t) = 0, \quad \text{for all } t \in [t_0, \infty),$$

under the assumption

$$\int_{t_0}^{\infty} \frac{t^{1+\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(t)} dt < \infty.$$

S. R. Grace et al. [65] have considered the oscillation of fourth-order delay differential equations

$$\left(r_3\left(r_2\left(r_1 y^{'}\right)^{'}\right)^{'}\right)^{'} + q(t)\, y(\tau(t)) = 0,$$

under the assumption

$$\int_{t_0}^{\infty} \frac{1}{r_{i(t)}} dt < \infty, \qquad i \in \{1, 2, 3\}.$$

Motivated by the papers mentioned above and other papers, here we wish to establish some new oscillation criteria for equation (2.1) which is considered a form that generalizes several differential equations and is similar to papers in a special case, for example, if $f(t,u) = 1$ and $g(t,u) = 0$, then equation (2.1) is reduced to the half-linear differential equations of fourth order with unbounded neutral coefficients

$$\left(a(t)\left(u^{(2)}(t)\right)^{\beta}\right)^{(2)} + \sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) = 0, \quad \text{for all } t \in [t_0, \infty). \tag{2.7}$$

If $a(t) = 1$ and $\beta = 1$, then equation (2.7) is reduced to the linear differential equations of fourth order with unbounded neutral coefficients

$$u^{(4)}(t) + \sum_{i=1}^{i=n} b_i(t)\, u(\tau_i(t)) = 0, \quad \text{for all } t \in [t_0, \infty), \tag{2.8}$$

which include several equations, the equation that has been studied by many authors [76, 37].

2.2 Oscillation Results

In this section, we establish some sufficient conditions which guarantee that every solution u of (2.1) oscillates on $[t_0, \infty)$.

For simplification, we consider the operator $P_{f,\beta}$ is defined by :

$$P_{f,\beta} u(t) = \frac{a(t)\left(u^{(2)}(t)\right)^{\beta}}{f(t, u(t))}, \quad \text{for } t \in [t_0, \infty).$$

and we note

$$\delta_+(t) = \max\{\delta(t), 0\}, \quad \text{for } t \in [t_0, \infty).$$

Theorem 2.2.1. *Suppose that the assumptions $(C_1)-(C_3)$ hold. Assume that there exist a positive function $\tau \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that for all sufficiently large $t_1 \in [t_0,\infty)$, for some $t_2 \in [t_1,\infty)$ ant $t_3 \in [t_2,\infty)$, such that*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\mathcal{A}_0(s,t_1,t_2)\,ds=\infty, \tag{2.9}$$

where

$$\mathcal{A}_0(s,t_1,t_2):=\tau(s)\,B_n(s)-\frac{1}{(\beta+1)^{\beta+1}}\frac{\left(\tau'_+(s)\right)^{\beta+1}}{\left(\varphi(s,t_1,t_2)\,\tau(s)\right)^{\beta}},\quad \text{for } t\in[t_3,\infty).$$

$$\varphi(t,t_1,t_2):=\int_{t_2}^{t}\left((s-t_1)\,\frac{\rho(s)}{a(s)}\right)ds,\quad \text{for } t\in[t_2,\infty).$$

If there exist a positive function $\theta \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, such that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\mathcal{A}_1(s,t_1)\,ds=\infty, \tag{2.10}$$

where

$$\mathcal{A}_1(s,t_1):=\theta(s)\psi^{\frac{1}{\beta}}(s)-\frac{\left\{\theta'(s)\right\}^2}{4\theta(s)},\quad \text{for } t\in[t_1,\infty).$$

$$\psi(t):=\frac{\rho(t)}{a(t)}\int_t^{\infty}\int_s^{\infty}B_n(\lambda)\,d\lambda ds,\quad \text{for } t\in[t_1,\infty).$$

Then any solution of (2.1) is oscillatory.

Proof.

Suppose that (2.1) has a nonoscillatory solution u on $[t_0,\infty)$. We may assume without loss of generality that there exists $t_1\in[t_0,\infty)$, such that

$$u(t)>0,\quad u^{\beta}(\tau_i(t))>0,\quad u(\eta(t))>0\quad \text{for } t\in[t_1,\infty),\ i\in\{1,...,n\}.$$

Since similar arguments can be made, for the case $u(t)<0$, eventually. Then u' is of constant sign eventually, that is to say, we have two cases. The first case if $u'(t)\geq 0$, for $t\in[t_1,\infty)$. From (2.5) and (2.6), we have

$$\sum\nolimits_{i=1}^{i=n}b_i(t)\,u^{\beta}(\tau_i(t))-\sigma(t)\,u^{\beta}(\eta(t))\geq B_n(t)\,u^{\beta}(t),\quad \text{for } t\in[t_1,\infty). \tag{2.11}$$

The second case if $u'(t)\leq 0$, for $t\in[t_1,\infty)$, by (2.5) and (2.6), we obtain

$$\sum\nolimits_{i=1}^{i=n}b_i(t)\,u^{\beta}(\tau_i(t))-\sigma(t)\,u^{\beta}(\eta(t))\geq B_n(t)\,u^{\beta}(\eta(t)),\quad \text{for } t\in[t_1,\infty).$$

Now, from (2.1), (2.11) and the above inequality, we obtain

$$\left(P_{f,\beta}u(t)\right)''\leq\sigma(t)\,u^{\beta}(\eta(t))-\sum\nolimits_{i=1}^{i=n}b_i(t)\,u^{\beta}(\tau_i(t))<0,\quad \text{for } t\in[t_1,\infty).$$

Thus, $t \to (P_{f,\beta}u(t))'$ is decreasing on $[t_1, \infty)$. We claim that $t \to P_{f,\beta}u(t) > 0$, for $t \in [t_1, \infty)$. If not, then there exist a $t_2 \in [t_1, \infty)$ and $m > 0$, such that

$$(P_{f,\beta}u(t))' \leq -m < 0, \quad \text{for } t \in [t_2, \infty).$$

Integrating the above inequality from t_2 to t, we obtain

$$P_{f,\beta}u(t) \leq -m(t - t_2) + c, \quad \text{for } t \in [t_2, \infty),$$

where $c := P_{f,\beta}u(t_2)$, we can choose $t_3 \in [t_2, \infty)$, such as

$$u''(t) \leq -\left(\frac{m}{2}\frac{t}{a(t)}f(t, u(t))\right)^{\frac{1}{\beta}} \leq -\left(\frac{mk}{2}\right)^{\frac{1}{\beta}}\left(\frac{t\rho(t)}{a(t)}\right)^{\frac{1}{\beta}}, \quad \text{for } t \in [t_3, \infty).$$

Integrating the above inequality from t_3 to t, we obtain

$$u'(t) \leq -\frac{mk}{2}\int_{t_2}^{t}\frac{s\rho(s)}{a(s)}ds + u(t_2), \quad \text{for } t \in [t_3, \infty).$$

which implies that $\lim_{t\to\infty} u'(t) = -\infty$. By lemma 1.2.3, we obtain $\lim_{t\to\infty} u(t) = -\infty$, which is a contradiction.

Then, there is $t_2 \geq t_1$, such that only one of the following two cases happens.

Case 1. Let $u''(t) > 0$, for all $t \in [t_1, \infty)$, then $u'(t) > 0$ for all $t \in [t_1, \infty)$, due to $(P_{f,\beta}u(t))' > 0$. Define the function ω by:

$$\omega_1(t) := \frac{\tau(t)}{u^\beta(t)}(P_{f,\beta}u(t))' > 0, \quad \text{for all } t \in [t_1, \infty).$$

Computing the derivative of ω_1 and from (2.1), we get

$$\begin{aligned}\omega_1'(t) &= \frac{\tau'(t)}{\tau(t)}\omega_1(t) - \frac{\tau(t)}{u^\beta(t)}\left(\sum_{i=1}^{i=n} b_i(t)\, u^\beta(\tau_i(t)) - g(t, u(\eta(t)))\right) \\ &\quad -\beta\frac{u'(t)}{u(t)}\omega_1(t).\end{aligned} \tag{2.12}$$

It follows from $u'(t) > 0$ for all $t \geq t_1$ and (2.6), (2.3) that

$$\sum_{i=1}^{i=n} b_i(t)\, u^\beta(\tau_i(t)) - g(t, u(\eta(t))) \geq B_n(t)\, u^\beta(t). \tag{2.13}$$

Therefore, $t \to (P_{f,\beta}u(t))'$ is a nonincreasing function on $[t_1, \infty)$. Then, we obtain

$$\begin{aligned}P_{f,\beta}u(t) &= \int_{t_1}^{t}(P_{f,\beta}u(t))'\,ds + P_{f,\beta}u(t_2) \\ &\geq (t - t_1)(P_{f,\beta}u(t))', \text{ for all } t \in [t_1, \infty).\end{aligned} \tag{2.14}$$

Hence,

$$\left(\frac{P_{f,\beta}u(t)}{t-t_1}\right)' = \frac{(P_{f,\beta}u(t))'}{t-t_1} - \frac{P_{f,\beta}u(t)}{(t-t_1)^2} \leq 0.$$

Thus, $t \to \dfrac{P_{f,\beta}u(t)}{t-t_1}$ is a nonincreasing function on $[t_2,\infty)$. Then, we obtain

$$\begin{aligned} u'(t) &\geq \int_{t_2}^{t} \frac{P_{f,\beta}u(t)}{s-t_1} \frac{f(s,u(s))(s-t_1)}{a(s)} ds \\ &\geq \frac{1}{f(t,u(t))}\left(\frac{a(t)}{t-t_1}\int_{t_2}^{t}\frac{\rho(s)(s-t_1)}{a(s)}ds\right)u''(t) \\ &\geq (P_{f,\beta}u(t))'\int_{t_2}^{t}\left((s-t_1)\frac{\rho(s)}{a(s)}\right)ds \\ &= \varphi(t,t_1,t_2)(P_{f,\beta}u(t))'. \end{aligned} \tag{2.15}$$

Substituting (2.15) and (2.13) in (2.12), we have

$$\begin{aligned} \omega_1'(t) &\leq \frac{\tau'(t)}{\tau(t)}\omega_1(t) - \tau(t)B_n(t) - \beta\frac{\omega_1(t)\varphi(t,t_1,t_2)}{u(t)}(P_{f,\beta}u(t))' \\ &\leq -\tau(t)B_n(t) + \frac{\tau_+'(t)}{\tau(t)}\omega_1(t) - \beta\frac{\varphi(t,t_1,t_2)}{\tau^{\frac{1}{\beta}}(t)}(\omega_1(t))^{1+\frac{1}{\beta}}. \end{aligned} \tag{2.16}$$

If we apply Lemma 1.2.5, we see that

$$\frac{\tau_+'(t)}{\tau(t)}\omega_1(t) - \beta\frac{\varphi(t,t_1,t_2)}{\tau^{\frac{1}{\beta}}(t)}(\omega_1(t))^{1+\frac{1}{\beta}} \leq \frac{1}{(\beta+1)^{\beta+1}}\frac{(\tau_+'(t))^{\beta+1}}{(\varphi(t,t_1,t_2)\tau(t))^{\beta}}. \tag{2.17}$$

Using (2.17) in (2.16), we obtain

$$\omega_1'(t) \leq -\tau(t)B_n(t) + \frac{1}{(\beta+1)^{\beta+1}}\frac{(\tau_+'(t))^{\beta+1}}{(\varphi(t,t_1,t_2)\tau(t))^{\beta}}.$$

Integrating the above inequality over $[t_3,t)$ yields

$$\int_{t_3}^{t}\left(\tau(s)B_n(t) - \frac{1}{(\beta+1)^{\beta+1}}\frac{(\tau_+'(s))^{\beta+1}}{(\varphi(s,t_1,t_2)\tau(s))^{\beta}}\right)ds \leq \omega_1(t_3),$$

which contradicts (2.9).

Case 2. Let $u''(t) < 0$, for all $t \in [t_1,\infty)$, then $u'(t) > 0$ for all $t \in [t_1,\infty)$, due to

$u(t) > 0$. Integrating (2.1) over $[t, s)$, we get

$$\begin{aligned}\int_t^s (P_{f,\beta} u(\tau))^{(2)} d\tau &= (P_{f,\beta} u(s))' - (P_{f,\beta} u(t))' \\ &\leq -\int_t^s \left(\sum_{i=1}^{i=n} b_i(\lambda) u^\beta(\tau_i(\lambda)) - g\left(\lambda, u^\beta(\eta(\lambda))\right)\right) d\lambda \\ &\leq -\int_t^s \left(\sum_{i=1}^{i=n} b_i(\lambda) u^\beta(\tau_i(\lambda)) - \sigma(\lambda) u^\beta(\eta(\lambda))\right) d\lambda.\end{aligned}$$

When s tends to ∞ in the above inequality, we obtain

$$(P_{f,\beta} u(t))' \geq \int_t^\infty \left(\sum_{i=1}^{i=n} b_i(\lambda) u^\beta(\tau_i(\lambda)) - \sigma(t) u^\beta(\eta(\lambda))\right) d\lambda.$$

It follows from $u'(t) > 0$, for all $t \in [t_1, \infty)$, and (2.13), we have

$$\begin{aligned}(P_{f,\beta} u(t))' &\geq \int_t^\infty B_n(s) u^\beta(s) ds \\ &\geq u^\beta(t) \int_t^\infty B_n(s) ds. \end{aligned} \tag{2.18}$$

Integrating above inequality over $[t, s)$, we get

$$P_{f,\beta} u(s) - P_{f,\beta} u(t) \geq \int_t^s \left(u^\beta(\rho) \int_\rho^\infty B_n(\lambda) d\lambda \right) d\rho$$

When s tends to ∞ in the above inequality, we obtain

$$-P_{f,\beta} u(t) \geq u^\beta(t) \int_t^\infty \int_s^\infty B_n(\lambda) d\lambda ds.$$

This means

$$-\left(\frac{u''(t)}{u(t)} \right)^\beta \geq \frac{\rho(t)}{a(t)} \int_t^\infty \int_s^\infty B_n(\lambda) d\lambda ds = \psi(t).$$

Then, we define the function ω_2 by:

$$\omega_2(t) := \theta(t) \frac{u'(t)}{u(t)} > 0, \quad \text{for all } t \in [t_1, \infty).$$

Computing the derivative of ω_2, we have

$$\begin{aligned}\omega_2'(t) &= \frac{\theta'(t)}{\theta(t)} \omega_2(t) + \theta'(t) \frac{u''(t)}{u(t)} - \theta'(t) \left| \frac{u'(t)}{u(t)} \right|^2 \\ &\leq \frac{\theta'(t)}{\theta(t)} \omega_2(t) - \theta(t) \psi^{\frac{1}{\beta}}(t) - \frac{\omega_2^2(t)}{\theta(t)} \\ &\leq -\theta(t) \psi^{\frac{1}{\beta}}(t) + \frac{1}{4} \frac{(\theta'(t))^2}{\theta(t)}.\end{aligned}$$

Integrating the above inequality from t_1 to t, we obtain

$$\int_{t_1}^{t}\left(\theta(s)\psi^{\frac{1}{\beta}}(s)-\frac{\left[\theta'(s)\right]^2}{4\theta(s)}\right)ds\leq\omega_2(t_1),$$

which contradicts (2.10).

This completes the proof. $\square$

Corollary 2.2.1. *Suppose that the assumptions $(C_1)-(C_3)$ hold, such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}B_n(s)\,ds=\infty, \tag{2.19}$$

where B_n is defined as in Theorem 2.2.1.

Then any solution of (2.1) is oscillatory.

Proof.

The proof is similar to that of Theorem 2.2.1, we put $\tau(t)=\theta(t)$ in Equations (2.9) and (2.10), we find Equations (2.19) and (2.32). $\square$

Theorem 2.2.2. *Suppose that the assumptions $(C_1)-(C_3)$ hold. Assume that there exist a positive function $\tau\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that for all sufficiently large $t_1\in[t_0,\infty)$, for some $t_2\in[t_1,\infty)$, and $t_3\in[t_2,\infty)$, such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_3}^{t}H(t,s)\left(\tau(s)B_n(s)-\frac{h^{\beta+1}(t,s)}{(\beta+1)^{\beta+1}\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}\right)ds = \infty, \tag{2.20}$$

where $\varphi(.,t_1,t_2)$ and B_n are defined as in Theorem 2.2.1.

If there exist a positive functions $\theta\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that (2.10) holds.

Then any solution of (2.1) is oscillatory.

Proof.

Suppose that (2.1) has a nonoscillatory solution u on $[t_0,\infty)$. We may assume without loss of generality that there exists $t_1\in[t_0,\infty)$, such that

$$u(t)>0,\quad u^{\beta}(\tau_i(t))>0,\quad u(\eta(t))>0\quad\text{for } t\in[t_1,\infty),\ i\in\{1,...,n\}.$$

Since similar arguments can be made, for the case $u(t)<0$, eventually. Then there are only the following two possible cases.

Case 1. If $u''(t)>0$ and $u'(t)>0$, for all $t\in[t_1,\infty)$. Multiplying both sides of (2.16) by $H(t,s)$, integrating it with respect to s from t_2 to t and using the propertie (1.1), we get

$$\begin{aligned}\int_{t_3}^{t} H(t,s)\tau(s)B_n(s)\,ds &\leq -\int_{t_3}^{t} H(t,s)\omega_1'(s)ds+\int_{t_3}^{t} H(t,s)\frac{\tau_+'(s)}{\tau(s)}\omega_1(s)ds\\ &\quad -\beta\int_{t_3}^{t} H(t,s)\frac{\varphi(s,t_1,t_2)}{\tau^{\frac{1}{\beta}}(s)}(\omega_1(s))^{1+\frac{1}{\beta}}ds\\ &\leq H(t,t_3)\omega_1(t_2)+\int_{t_3}^{t}\left(\frac{h(t,s)}{\tau(s)}H(t,s)\right)\omega_1(s)ds\\ &\quad -\beta\int_{t_3}^{t} H(t,s)\frac{\varphi(s,t_1,t_2)}{\tau^{\frac{1}{\beta}}(s)}(\omega_1(s))^{1+\frac{1}{\beta}}ds.\end{aligned}$$

If we apply Lemma 1.2.5, we see that

$$\begin{aligned}\int_{t_3}^{t} H(t,s)\tau(s)B_n(s)\,ds &\leq H(t,t_2)\omega_1(t_2)\\ &\quad +\int_{t_3}^{t}\frac{H(t,s)}{(\beta+1)^{\beta+1}}\frac{h^{\beta+1}(t,s)}{\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}ds.\end{aligned}$$

which implies that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t} H(t,s)\left(\tau(s)B_n(s)-\frac{1}{(\beta+1)^{\beta+1}}\frac{h^{\beta+1}(t,s)}{\varphi^{\beta}(s,t_1,t_2)\tau^{\beta}(s)}\right)ds\leq\omega_1(t_3),$$

which contradicts (2.20).

The proof of **case (2)** is the same as that of **case (2)** in Theorem 2.2.1, and so is omitted. This completes the proof. □

As a Theorem of the previous result, we deduce the following Corollarie.

Corollary 2.2.2. *Suppose that the assumptions $(C_1)-(C_3)$ hold. Assume that there exist $m\in\mathbb{N}$ such that, for all sufficiently large $t_1\in[t_0,\infty)$, for some $t_2\in[t_1,\infty)$, and $t_3\in[t_2,\infty)$, such that*

$$\limsup_{t\to\infty} t^{-m}\int_{t_3}^{t}(t-s)^m B_n(s)-\left(\frac{n}{\beta+1}\right)^{\beta+1}\frac{(t-s)^{-(\beta+1)}}{\varphi^{\beta}(s,t_1,t_2)}ds=\infty, \tag{2.21}$$

where $\varphi(.,t_1,t_2)$ and B_n are defined as in Theorem 2.2.1.
If there exist a positive functions $\theta\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that (2.10) holds.
Then any solution of (2.1) is oscillatory.

Proof.

The proof is similar to that of Theorem 2.2.1, we put $\tau(t) = 1$ and $H(t,s) = (t-s)^m$, for $t > s > t_0$ in Equation (2.20), we find Equation (2.21). $\square$

Theorem 2.2.3. *Assume that conditions* (C_1)-(C_3) *holds. Assume that there exist a positive function* $\varphi \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$, *such that for all sufficiently large* $t_1 \in [t_0,\infty)$, *for some* $t_2 \in [t_1,\infty)$, *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(B_n(s) \int_{t_1}^{s} \Lambda(\rho)\, d\rho \right) ds = \infty, \tag{2.22}$$

and

$$\varphi(t) - \varphi'(t)(t-t_1) \leq 0, \quad \text{for all } t \in [t_2,\infty). \tag{2.23}$$

where

$$\Lambda(t) := \left((t-t_1) \int_t^{\infty} B_n(s)\, ds \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s)\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds.$$

If there exist a positive functions $\theta \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$ *such that* (2.10) *holds.*

Then any solution of (2.1) *is oscillatory.*

Proof.

Suppose that (2.1) has a nonoscillatory solution u on $[t_0,\infty)$. We may assume without loss of generality that there exists $t_1 \in [t_0,\infty)$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in [t_1,\infty),\ i \in \{1,...,n\}.$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then there are only the following two possible cases.

Case 1. If $u''(t) > 0$ and $u'(t) > 0$, for all $t \in [t_1,\infty)$. Using (2.1), it follows from (2.14) that

$$\left(\frac{P_{f,\beta}u(t)}{\varphi(t)} \right)' \leq \frac{P_{f,\beta}u(t)}{\varphi^2(t)(t-t_1)} \left(\varphi(t) - \varphi'(t)(t-t_1) \right) \leq 0.$$

Thus, $t \to \dfrac{P_{f,\beta}u(t)}{\varphi(t)}$ is a nonincreasing function on $[t_1,\infty)$. Then,

$$\begin{aligned} u'(t) &\geq \left(\frac{P_{f,\beta}u(t)}{\varphi(t)} \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s) f(t,u(s))}{a(s)} \right)^{\frac{1}{\beta}} ds \\ &\geq \left(\frac{P_{f,\beta}u(t)}{\varphi(t)} \right)^{\frac{1}{\beta}} \int_{t_1}^{t} \left(\frac{\varphi(s)\rho(s)}{a(s)} \right)^{\frac{1}{\beta}} ds. \end{aligned} \tag{2.24}$$

It follows from (2.14) and (2.18) that

$$\begin{aligned} u^{'}(t) &\geq u(t)\left((t-t_1)\int_t^{\infty} B_n(s)\,ds\right)^{\frac{1}{\beta}}\int_{t_1}^{t}\left(\frac{\varphi(s)\rho(s)}{a(s)}\right)^{\frac{1}{\beta}} ds \\ &= \Lambda(t)\,u(t)\,, \text{ for all } t\in[t_1,\infty). \end{aligned}$$

Clearly $u^{'}(t)>0$, for $t\in[t_1,\infty)$, then there exists $\ell>0$, such that

$$u(t)\geq \ell\int_{t_1}^{t}\Lambda(s)\,ds, \text{ for all } t\in[t_1,\infty).$$

Using (2.1), (2.5), and the above inequality, we obtain

$$(P_{f,\beta}u(t))^{(2)}\leq -\ell B_n(t)\int_{t_1}^{t}\Lambda(s)\,ds.$$

Integrating the above inequality over $[t_1,t)$, we obtain

$$(P_{f,\beta}u(t))^{'}\leq (P_{f,\beta}u(t_1))^{'}-\ell\int_{t_1}^{t}\left(B_n(s)\int_{t_1}^{s}\Lambda(\rho)\,d\rho\right)ds.$$

By (2.10), this gives

$$\liminf_{t\to\infty}(P_{f,\beta}u(t))^{'}=-\infty.$$

If we apply Lemma 1.2.3, give us $\lim_{t\to\infty}P_{f,\beta}u(t)=-\infty$, which is a contradiction.

The proof of **case (2)** is the same as that of **case (2)** in Theorem 2.2.1, and so is omitted.

This completes the proof. □

Theorem 2.2.4. *Assume that conditions (C_1)-(C_3) holds. Assume that there exist a positive functions $\varphi,\xi\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$, such that for all sufficiently large $t_1\in[t_0,\infty)$, for some $t_2\in[t_1,\infty)$, such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\frac{\beta\varphi(s)\xi(s)}{\pi(s)(s-t_1)^{\beta+1}}-\frac{\xi(s)}{(s-t_1)^{\beta+1}\pi^{\beta}(s)}-\frac{\xi^{'}(s)\varphi(s)}{\pi(s)(s-t_1)^{\beta}}\right)ds=\infty, \tag{2.25}$$

where φ si defined as in Theorem 2.2.3, and

$$\xi(t)+(t-t_1)\,\pi^{\beta-1}(t)\,\xi^{'}(t)\varphi(t)\leq\beta\pi^{\beta-1}(t)\,\varphi(t)\,\xi(t),\ \ \textit{for all}\ t\in[t_2,\infty).$$

$$\pi(t):=\int_{t_1}^{t}\left(\frac{\varphi(s)\rho(s)}{a(s)}\right)^{\frac{1}{\beta}}ds,\ \ \textit{for all}\ t\in[t_2,\infty).$$

If there exist a positive functions $\theta\in\mathcal{C}^1([t_0,\infty),\mathbb{R})$ such that (2.10) holds.

Then any solution of (2.1) is oscillatory.

Proof.

Suppose that (2.1) has a nonoscillatory solution u on $[t_0,\infty)$. We may assume without loss of generality that there exists $t_1 \in [t_0,\infty)$, such that

$$u(t) > 0, \quad u^{\beta}(\tau_i(t)) > 0, \quad u(\eta(t)) > 0 \quad \text{for } t \in [t_1,\infty),$$

Since similar arguments can be made, for the case $u(t) < 0$, eventually. Then there are only the following two possible cases.

Case 1. If $u''(t) > 0$ and $u'(t) > 0$, for all $t \in [t_1,\infty)$. We define the function ω_3 by:

$$\omega_3(t) := \frac{\xi(t)}{u^{\beta}(t)} P_{f,\beta} u(t) > 0, \quad \text{for all } t \in [t_1,\infty).$$

Using (2.24) and (2.14), we arrive at

$$\frac{u'(t)}{u(t)} \geq \left(\frac{\pi(t)}{\varphi(t)\,\xi(t)}\right)^{\frac{1}{\beta}} \omega_3^{\frac{1}{\beta}}(t), \quad \text{for all } t \in [t_1,\infty). \tag{2.26}$$

$$\left(P_{f,\beta} u(t)\right)' \leq \frac{\left(u'(t)\right)^{\beta}}{(t-t_1)\,\pi^{\beta}(t)}, \quad \text{for all } t \in [t_1,\infty). \tag{2.27}$$

It follows from Corollary 1.2.1, we have

$$u(t) \geq u'(t)\,(t-t_1), \quad \text{for all } t \in [t_1,\infty).$$

This implies that

$$\omega_3(t) \leq \frac{\varphi(t)\,\xi(t)}{\pi(t)\,(t-t_1)^{\beta}}, \quad \text{for all } t \in [t_2,\infty). \tag{2.28}$$

By (2.27) and as above inequality, we get

$$\frac{\left(P_{f,\beta} u(t)\right)'}{u^{\beta}(t)} \leq \frac{1}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)}, \quad \text{for all } t \in [t_2,\infty). \tag{2.29}$$

Computing the derivative of ω_3, we have

$$\omega_3'(t) = \frac{\xi'(t)}{\xi(t)}\omega_3(t) + \frac{\xi(t)}{u^{\beta}(t)}\left(P_{f,\beta} u(t)\right)' - \beta\omega_3(t)\frac{u'(t)}{u(t)}.$$

Substituting (2.29), (2.28) and (2.26) in the above equality, we obtain

$$\begin{aligned}
\omega_3'(t) &\leq \frac{\xi(t)}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)} + \frac{\xi'(t)}{\xi(t)}\omega_3(t) - \beta\left(\frac{\pi(t)}{\varphi(t)\,\xi(t)}\right)^{\frac{1}{\beta}} \omega_3^{1+\frac{1}{\beta}}(t) \\
&\leq \frac{\xi(t)}{(t-t_1)^{\beta+1}\,\pi^{\beta}(t)} + \frac{\xi'(t)\varphi(t)}{\pi(t)\,(t-t_1)^{\beta}} - \frac{\beta\varphi(t)\,\xi(t)}{\pi(t)\,(t-t_1)^{\beta+1}} \leq 0.
\end{aligned}$$

Integrating the above inequality over $[t_2, t)$, we obtain

$$\int_{t_2}^{t}\left(\frac{\beta\varphi\left(s\right)\xi(s)}{\pi\left(s\right)\left(s-t_1\right)^{\beta+1}}-\frac{\xi(s)}{\left(s-t_1\right)^{\beta+1}\pi^{\beta}\left(s\right)}-\frac{\xi^{'}(s)\varphi\left(s\right)}{\pi\left(s\right)\left(s-t_1\right)^{\beta}}\right)ds\leq\omega_3\left(t_2\right),$$

which contradicts (2.25).

The proof of **case (2)** is the same as that of **case (2)** in Theorem 2.2.1, and so is omitted. This completes the proof. □

Let $\xi\left(t\right)=t-t_1$, for $t\in[t_2,\infty)$. Then Theorem 2.2.4 yields the following result.

Corollary 2.2.3. *Assume that conditions (C_1)-(C_3) holds. Assume that there exist a positive functions $\varphi\in\mathcal{C}^1\left([t_0,\infty),\mathbb{R}\right)$, such that for all sufficiently large $t_1\in[t_0,\infty)$, for some $t_2\in[t_1,\infty)$, such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\frac{\beta\varphi\left(s\right)\pi^{\beta-1}\left(s\right)-1}{\pi^{\beta}\left(s\right)\left(s-t_1\right)^{\beta}}ds=\infty,$$

where φ and π are defined as in Theorem 2.2.4, and

$$\pi^{\beta-1}\left(t\right)\varphi\left(t\right)\geq\frac{1}{\beta},\quad \text{for all } t\in[t_2,\infty).$$

If there exist a positive functions $\theta\in\mathcal{C}^1\left([t_0,\infty),\mathbb{R}\right)$ such that (2.10) holds. Then any solution of (2.1) is oscillatory.

2.3 Examples and Discussions

In this section, we give an example where we apply Theorems 2.2.1 and 2.2.3 to some particular cases.

Example 4. *Consider the neutral differential equation*

$$\left(\frac{u^{(2)}\left(t\right)}{e^{t}}\right)^{(2)}+\sum_{i=1}^{i=n}e^{-t}u\left(t-i\right)=0,\quad \text{for all } t\geq n+1. \tag{2.30}$$

Here, $\beta=1$, $r\left(t\right)=1$, $n\in\mathbb{N}$, $a\left(t\right)=1$, $\tau_i\left(t\right)=t-i$, $b_i\left(t\right)=e^{-t}$, for all $i\in\{1,2,...,n\}$, $f\left(t,u\right)=e^{t}$ and $g\left(t,u\right)=0$. Then $\rho\left(t\right)=e^{t}$, $\sigma\left(t\right)=0$ and the hypotheses $(C_1)-(C_4)$ are holds.

On the other hand, we see that

$$B_n\left(t\right)=\sum_{i=1}^{i=n}b_i\left(t\right)-\sigma\left(t\right)=ne^{-t},\quad \text{for all } t\geq n+1.$$

$$\varphi(t, t_1, t_2) \approx \frac{t}{2} e^t, \quad \textit{for } t \textit{ large enough,}$$

$$\psi(t) = n, \quad \textit{for } t \textit{ large enough.}$$

Let $\tau(t) = e^t$ *and* $\theta(t) = 1$, *for all* $t \geq n+1$, *then, we get*

$$\mathcal{A}_0(t, t_1, t_2) = n - \frac{1}{4}\frac{e^t}{\varphi(t, t_1, t_2)} \approx n - \frac{1}{4t}, \quad \textit{for } t \textit{ large enough,}$$

$$\mathcal{A}_1(t, t_1) = n, \quad \textit{for all } t \geq t_2$$

Thus, (2.9) *and* (2.10) *holds.*

By Theorem 2.2.1, equation (2.30) *is oscillatory.*

Example 5. *Consider the hybrid differential equation*

$$\left(e^{-u^2-t}\sqrt[3]{u^{(2)}(t)}\right)^{(2)} + e^{-t}\sqrt[3]{u(t)} = 0, \quad \textit{for all } t \geq 0, \tag{2.31}$$

Here, $\beta = \frac{1}{3}$, $a(t) = 1$, $n = 1$, $b_1(t) = e^{-t}$, $\tau_1(t) = t$, $f(t,u) = e^{u^2+t}$, *and* $g(t,u) = 0$. *Then* $\rho(t) = e^t$, $\sigma(t) = 0$ *and the hypotheses* (C_1)-(C_4) *are holds. Let* $\varphi(t) = e^{3t}$, *for* $t \geq 0$, *then* (2.23) *holds,*

$$\Lambda(t) \geq d\sqrt[3]{t}e^t, \quad \textit{for } t \textit{ large enough,}$$

where $d > 0$, *then* (2.22) *holds. Therefore, we have*

$$\psi(t) = 1, \quad \textit{for } t \textit{ large enough,}$$

Let $\theta(t) = 1$, *for all* $t \geq 0$. *Thus,* (2.10) *holds.*

By Theorem 2.2.3, equation (2.31) *is oscillatory.*

Remark. *These results show that the coefficient functions* $\{b_i\}_{i\in\{1,..,n\}}$ *plays an important role in oscillation of fourth-order hybrid nonlinear dynamic equation; see the details in Example 4 and differences between Corollary 2.2.1 and Theorem 2.2.1, Theorem 2.2.2, Theorem 2.2.3.*

Remark. *If we consider a fourth-order hybrid nonlinear functional dynamic equations with damping on time scale*

$$\left(\frac{a(t)\left(u^{\Delta^2}(t)\right)^{\beta}}{f(t, u(t))}\right)^{\Delta^2} + \sum_{i=1}^{i=n} b_i(t)\, u^{\beta}(\tau_i(t)) = g(t, u(\eta(t))), \tag{2.32}$$

on an arbitrary time scale $\mathbb{T}$ *with* $\sup \mathbb{T} = \infty$*. Thus, equation* (2.1) *becomes a special case of equation* (2.32) *in a case* $\mathbb{T} = \mathbb{R}$*. From the method given in this chapter, one can obtain some oscillation criteria for* (3.20)*. It means obtaining generalizations of the Theorems 2.2.1, 2.2.2, 2.2.3 and 2.2.4. The details are left to the reader.*

Chapter 3

Oscillation theorems for advanced differential equations

In this chapter, we use the recursive sequence we have constructed establish some new oscillation results of first-order linear dynamic equations with damping. The original results of this chapter are published in [4].

3.1 Introduction

In this chapter, we consider the advanced differential equation of the form

$$u^{'}(t) - \sum\nolimits_{i=1}^{i=k} q_i(t)\, u^{\alpha}(\tau_i(t)) = 0, \quad \text{for } t \in [t_0, \infty) \tag{3.1}$$

where k is an integer and α is a quotient of odd integer, such that $k \geq 1$ and $\alpha \geq 1$. The functions $\{q_i\}_{i\in\{1,\ldots,k\}}$, $\{\tau_i\}_{i\in\{1,\ldots,k\}}$ are continuous and positive that satisfy the conditions stated below:

$(\mathcal{H}_1)$ $\{\tau_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{C}([t_0,\infty),[t_0,\infty))$ satisfy

$$\tau_i(t) \geq t, \qquad \text{for } t \in [t_0, \infty).$$

and

$$\lim_{t\to\infty} \tau_i(t) = \infty, \qquad \text{for } i \in \{1, 2, \ldots, k\},$$

$(\mathcal{H}_2)$ $\{q_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{C}([t_0,\infty),[0,\infty))$, such that $Q := \sum_{i=1}^{i=k} q_i \neq 0$ on any interval of the form $[t_0,\infty)$ and $\displaystyle\int_t^{\tau(t)} Q(s)\, ds$ increased on $[t_0,\infty)$, where

$$\tau(t) := \min\{\tau_i(t) : i \in \{1, ..k\}\}, \qquad \text{for } t \in [t_0, \infty).$$

By a solution of (3.1) we mean a nontrivial real-valued function $u \in \mathcal{C}^1([T_u, \infty), \mathbb{R})$, $T_u \in [t_0, \infty)$ which satisfies (3.1) on $[T_u, \infty)$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (3.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (3.1) is called oscillatory if all its solutions are oscillatory.

3.2 Oscillation Results

To derive oscillation results in this section, we need the following lemmas.

Definition 3.2.1. *Let us define a sequence of functions by the recurrence relation*

$$J_{n+1}(t) := \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s) \exp(J_n(t))\, ds, \quad \text{for } t \in [t_0, \infty), \tag{3.2}$$

with

$$J_0(t) := \sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds, \quad \text{for } t \in [t_0, \infty), \tag{3.3}$$

Lemma 3.2.1. *Assume $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold and $\alpha = 1$. If u is a positive solution of (3.1), then the sequence $\{J_n(t) : n \in \mathbb{N}\}$ converges.*

Proof.
Let u be an eventually positive solution of (3.1). From (3.1), we have $u'(t) \geq 0$, for $t \in [t_0, \infty)$.
On the other hand, for $i \in \{1, ..., k\}$, we have

$$\begin{aligned}
\ln\left(\frac{u(\tau_i(t))}{u(t)}\right) &= \int_t^{\tau_i(t)} \frac{u'(s)}{u(s)} ds \\
&= \sum_{m=1}^{m=k} \int_t^{\tau_i(t)} q_m(s) \frac{u(\tau_m(s))}{u(s)} ds \\
&\geq \sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s) \frac{u(\tau_m(s))}{u(s)} ds \qquad (3.4)\\
&\geq \sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s)\, ds \geq J_0(t), \quad \text{for } t \in [t_0, \infty).
\end{aligned}$$

This means,

$$\frac{u(\tau_i(t))}{u(t)} \geq \exp(J_0(t)), \quad \text{for } t \in [t_0, \infty) \text{ and for } i \in \{1, ..., k\}.$$

It follows from (3.4) and the above inequality, we obtain

$$\begin{aligned}\ln\left(\frac{u(\tau_i(t))}{u(t)}\right) &\geq \sum_{m=1}^{m=k}\int_t^{\tau(t)} q_m(s)\exp(J_0(s))\,ds\\ &= J_1(t), \quad \text{for } t\in[t_0,\infty).\end{aligned}$$

By induction, we can see that if

$$\ln\left(\frac{u(\tau_i(t))}{u(t)}\right) \geq J_n(t), \quad \text{for } t\in[t_0,\infty) \text{ and for } i\in\{1,...,k\}.$$

In the same way, we find that the inequality is true for $n+1$. By (3.2) and the above inequality, we conclude that the sequence $\{J_n(t) : n\in\mathbb{N}\}$ is increasing, thus $\{J_n(t) : n\in\mathbb{N}\}$ is converges. □

Lemma 3.2.2. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha=1$. *The sequence* $\{J_n(t) : n\in\mathbb{N}\}$ *defined by* (3.2), *converges if and only if*

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds \leq \frac{1}{e}, \quad \textit{for all } t\in[t_0,\infty). \tag{3.5}$$

Proof.

Sufficient: Suppose that (3.3) is true. Then

$$J_0(t)\leq\frac{1}{e}=v_0, \quad \text{for all } t\in[t_0,\infty),$$

Then, we get

$$\begin{aligned}J_1(t) &\leq \sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\exp(J_0(t))\,ds\\ &\leq \sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\,ds\exp(v_0)\\ &\leq v_0\exp(v_0)=v_1.\end{aligned}$$

By induction, we can see that if

$$J_n(t)\leq v_0\exp(v_n)<1.$$

In view of Lemma 1.2.4, $\{J_n(t) : n\in\mathbb{N}\}$ converges.

Necessary: Suppose that $\{J_n(t) : n\in\mathbb{N}\}$ converges, then there is a positive real function denoted $J(t)$, such that $J(t)=\lim_{n\to\infty} J_n(t)$, by (3.2), we find that the function J is satisfied

$$J(t)=\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i(s)\exp(J(s))\,ds, \quad \text{for } t\in[t_0,\infty). \tag{3.6}$$

By the hypothesis, we have that the function J_0 is an increasing on $[t_0, \infty)$, then by induction deduce that functions J_n are increasing on $[t_0, \infty)$, we conclude that the function J is increased on $[t_0, \infty)$. By the above equality, we obtain

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \leq J(t) \exp(-J(t)), \quad \text{for } t \in [t_0, \infty).$$

On the other hand, we have

$$\max\{x \exp(-x) : x \geq 1\} = \frac{1}{e}.$$

By (3.6), deduce that

$$J(t) \geq 1, \qquad \text{for } t \in [t_0, \infty).$$

From the above, we deduce

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds \leq \frac{1}{e}, \quad \text{for } t \in [t_0, \infty).$$

This completes the proof. □

Remark. *Assume* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold and* $\alpha = 1$. *If* u *is an positive solution of* (3.1), *then inequality* (3.5) *is satisfied.*

Next, we consider the advanced differential equation (3.1) subject to the initial condition

$$u(t_0) := a > 0. \tag{3.7}$$

Definition 3.2.2. *Let us define a sequence of functions by the recurrence relation*

$$I_{n+1}^{\alpha}(t) := \left(1 + a^{\alpha-1}(\alpha-1)\sum_{i=1}^{i=k} \int_t^{\tau_i(t)} q_i(s)\, I_n^{\alpha}(s)\, ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \textit{for } t \in [t_0, \infty), \tag{3.8}$$

with

$$I_0^{\alpha}(t) := \left(1 + a^{\alpha-1}(\alpha-1)\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \textit{for } t \in [t_0, \infty), \tag{3.9}$$

where $\alpha > 1$.

Lemma 3.2.3. *Assume* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold and* $\alpha > 1$. *If* u *is an positive solution of* (3.1), *then the sequence* $\{I_n^{\alpha}(t) : n \in \mathbb{N}\}$ *converges.*

Proof.

Let u be an eventually positive solution of (3.1). From (3.1), we have $u^{'}(t) \geq 0$, for $t \in [t_0, \infty)$. On the other hand, for $i \in \{1, ..., k\}$, we have

$$\begin{aligned}
\frac{1}{u^{\alpha-1}(t)} - \frac{1}{u^{\alpha-1}(\tau_i(t))} &= (\alpha-1)\int_t^{\tau_i(t)} \frac{u^{'}(s)}{u^{\alpha}(s)} ds \\
&= (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau_i(t)} q_m(s) \frac{u^{\alpha}(\tau_m(s))}{u^{\alpha}(s)} ds \\
&\geq (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s) \frac{u^{\alpha}(\tau_m(s))}{u^{\alpha}(s)} ds \qquad (3.10) \\
&> (\alpha-1)\sum_{m=1}^{m=k} \int_t^{\tau(t)} q_m(s)\, ds, \text{ for all } t \in [t_0, \infty). (3.11)
\end{aligned}$$

Since u is increasing on $[t_0, \infty)$, then

$$u(t) \geq u(t_0) = a, \quad \text{for all } t \in [t_0, \infty).$$

Hence

$$\frac{u^{\alpha-1}(\tau_i(t))}{u^{\alpha-1}(t)} \geq 1 + a^{\alpha-1}\left(\frac{1}{u^{\alpha-1}(t)} - \frac{1}{u^{\alpha-1}(\tau_i(t))}\right), \quad \text{for all } t \in [t_0, \infty). \qquad (3.12)$$

It follows from (3.11) and the above inequality, we obtain

$$\begin{aligned}
\frac{u^{\alpha}(\tau_i(t))}{u^{\alpha}(t)} &\geq \left(1 + a^{\alpha-1}(\alpha-1)\sum_{m=1}^{m=k}\int_t^{\tau(t)} q_m(s)\, ds\right)^{\frac{\alpha}{\alpha-1}} \\
&= I_0^{\alpha}(t), \quad \text{for all } t \in [t_0, \infty).
\end{aligned}$$

It follows from (3.10), (3.12) and the above inequality, we obtain

$$\frac{u^{\alpha-1}(\tau_i(t))}{u^{\alpha-1}(t)} \geq 1 + a^{\alpha-1}(\alpha-1)\sum_{m=1}^{m=k}\int_t^{\tau(t)} q_m(s)\, I_0^{\alpha}(s)\, ds, \quad \text{for all } t \in [t_0, \infty),$$

or

$$\begin{aligned}
\frac{u^{\alpha}(\tau_i(t))}{u^{\alpha}(t)} &\geq \left(1 + a^{\alpha-1}(\alpha-1)\sum_{m=1}^{m=k}\int_t^{\tau(t)} q_m(s)\, I_0^{\alpha}(t)\, ds\right)^{\frac{\alpha}{\alpha-1}} \\
&= I_1^{\alpha}(t), \quad \text{for } t \in [t_0, \infty).
\end{aligned}$$

By induction, we can see that if

$$\frac{u^{\alpha}\left(\tau_i\left(t\right)\right)}{u^{\alpha}\left(t\right)} \geq I_n^{\alpha}\left(t\right), \quad \text{for } t \in [t_0, \infty) \text{ and for } i \in \{1, ..., k\}.$$

In the same way, we find that the inequality is true for $n+1$.

We conclude that the sequence $\{I_n^{a}\left(t\right) : n \in \mathbb{N}\}$ is increasing and increased, then $\{I_n^{\alpha}\left(t\right) : n \in \mathbb{N}\}$ is converges. □

Lemma 3.2.4. *The sequence $\{I_n^{\alpha}\left(t\right) : n \in \mathbb{N}\}$ defined by (3.8), converges if and only if*

$$\sum_{i=1}^{i=k} \int_t^{\tau(t)} q_i\left(s\right) ds \leq \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}, \tag{3.13}$$

where $\alpha > 1$.

Proof.

Suppose that $\{I_n^{\alpha}\left(t\right) : n \in \mathbb{N}\}$ converges. Then there is a positive real function denoted $I^{\alpha}\left(t\right)$, such that $I^{\alpha}\left(t\right) = \lim_{n\to\infty} I_n^{\alpha}\left(t\right)$, by (3.8), we find that the function I^{α} is satisfied

$$I^{\alpha}\left(t\right) = \left(1 + a^{\alpha-1}\left(\alpha-1\right)\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right) I^{\alpha}\left(s\right) ds\right)^{\frac{\alpha}{\alpha-1}}, \quad \text{for } t \in [t_0, \infty). \tag{3.14}$$

By the hypothesis, we have that the function I_0^{α} is an increasing on $[t_0, \infty)$, then by induction deduce that functions I_n^{α} are increasing on $[t_0, \infty)$, we conclude that the function I^{α} is increased on $[t_0, \infty)$. By the above equality, we obtain

$$a^{\alpha-1}\left(\alpha-1\right)\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right) ds \leq \frac{\left(I^{\alpha}\left(t\right)\right)^{1-\frac{1}{\alpha}} - 1}{I^{\alpha}\left(t\right)}, \quad \text{for } t \in [t_0, \infty).$$

On the other hand, we have

$$\sup\left\{\frac{x^{1-\frac{1}{\alpha}} - 1}{x} : x \geq 1\right\} = \frac{\alpha-1}{\alpha^{\frac{\alpha}{\alpha-1}}},$$

By (3.14), deduce that $I^{\alpha}\left(t\right) \geq 1$, for $t \in [t_0, \infty)$, which means that

$$\sum_{i=1}^{i=k}\int_t^{\tau(t)} q_i\left(s\right) ds \leq \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}, \quad \text{for } t \in [t_0, \infty).$$

This completes the proof. □

Now, we establish some sufficient conditions which guarantee that every solution u of (3.1) oscillates on $[t_0, \infty)$.

Theorem 3.2.1. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha = 1$. *For all sufficiently large* $t_1 \geq t_0$, *such that*

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\,ds > \frac{1}{e}, \quad \text{for } t \geq t_1. \tag{3.15}$$

Then any solution of (3.1) *is oscillatory.*

Proof.
Suppose that (3.1) has a nonoscillatory solution u on $[t_0, \infty)$. Since $-u$ is also a solution of (3.1), we can confine our discussion only to the case where the solution u is eventually positive solution of (3.1). We may assume without loss of generality that there exists $t_1 \geq t_0$, such that

$$u(t) > 0 \quad \text{and} \quad u(\tau_i(t)) > 0, \text{ for all } t \geq t_1 \text{ and } i \in \{1, 2, ..., k\}.$$

This means the following equation (3.1) has a positive solution u on $[t_1, \infty)$.

$$u^{'}(t) - \sum\nolimits_{i=1}^{i=k} q_i(t)\, u(\tau_i(t)) = 0, \quad \text{for } t \geq t_1$$

By Lemma 3.2.1 and Lemma 3.2.2, we obtain

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\,ds \leq \frac{1}{e}, \quad \text{for } t \geq t_1.$$

which contradicts (3.15).
This completes the proof. □

As a Theorem of the previous result, we deduce the following corollaries.

Corollary 3.2.1. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha = 1$, *such that*

$$\liminf_{t\to\infty} \sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\,ds > \frac{1}{e}.$$

Then any solution of (3.1) *is oscillatory.*

Corollary 3.2.2. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha = 1$, *such that*

$$\limsup_{t\to\infty} \sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\,ds > 1.$$

Then any solution of (3.1) *is oscillatory.*

Theorem 3.2.2. *Assume* $(\mathcal{H}_1)-(\mathcal{H}_2)$ *hold and* $\alpha > 1$. *For all sufficiently large* $t_1 \geq t_0$, *such that*

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\,ds > \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}, \quad \text{for } t \geq t_1. \tag{3.16}$$

Then any solution of (3.1)-(3.7) *is oscillatory.*

Proof.

Suppose that (3.1) has a nonoscillatory solution u on $[t_0, \infty)$. Since $-u$ is also a solution of (3.1), we can confine our discussion only to the case where the solution u is eventually positive solution of (3.1). We may assume without loss of generality that there exists $t_1 \geq t_0$, such that

$$u\left(t\right) > 0 \quad \text{and} \quad u\left(\tau_i\left(t\right)\right) > 0, \text{ for all } t \geq t_1 \text{ and } i \in \left\{1, 2, ..., k\right\}.$$

By Lemma 3.2.3 and Lemma 3.2.4, we obtain

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i\left(s\right) ds \leq \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}, \quad \text{for } t \in [t_0, \infty).$$

which contradicts (3.16).

This completes the proof. □

As a Theorem of the previous result, we deduce the following corollarie.

Corollary 3.2.3. *Assume* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold and* $\alpha > 1$*, such that*

$$\liminf_{t\to\infty} \sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i\left(s\right) ds > \frac{a^{1-\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}.$$

Then any solution of (3.1)-(3.7) *is oscillatory.*

3.3 Examples

Next, we give an example where we apply Theorems 3.2.1 and 3.2.2 to some particular cases.

Example 6. *Consider the delay differential equation*

$$x^{'}\left(t\right) - \sum\nolimits_{i=1}^{i=k} x\left(t+i\right) = 0, \quad \textit{for all } t \geq 0. \tag{3.17}$$

Here, $k \in \mathbb{N}^*$*,* $\alpha = 1$*,* $q_i\left(t\right) = 1$*,* $\tau_i\left(t\right) = t + i > t$*, for all* $i \in \left\{1, 2, ..., n\right\}$*, and* $\tau\left(t\right) = t + 1$*. Then* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *holds. On the other hand, we have*

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i\left(s\right) ds = \frac{k}{2}\left(k+1\right) > \frac{1}{e}, \quad \textit{for all } t \geq 0.$$

Thus, (3.15) *holds.*

By Theorem 3.2.1, equation (3.17) *is oscillatory.*

Example 7. *Consider the delay differential equation*

$$x^{'}(t) - tx^{3}(t+1) = 0, \quad \text{for all } t \geq 0. \tag{3.18}$$

subject to the initial condition

$$u(0) = a \geq 0. \tag{3.19}$$

Here, $k = 1$*,* $\alpha = 3 > 1$*,* $q_1(t) = t$*, and* $\tau(t) = \tau_1(t) = t + 1 > t$*.*

Then $(\mathcal{H}_1) - (\mathcal{H}_2)$ *holds. On the other hand, we have*

$$\int_t^{\tau(t)} q(s)\, ds = \frac{1}{2}(2t+1) \geq \frac{1}{2}, \quad \text{for all } t \geq 0.$$

If $u(0) = a > 0.620$*, then* (3.16) *holds.*

By Theorem 3.2.2, equation (3.18)-(3.19) *is oscillatory.*

3.4 Conclusion

Remark. *For* $\alpha > 1$*, we pose* $\psi_a(\alpha) = a^{1-\alpha}\alpha^{\frac{\alpha}{1-\alpha}}$*, we have* $\lim_{\alpha \to a^{+}} \psi_a(\alpha) = \frac{1}{e} = \psi_a(1)$*, then, we can summarize the two conditions* (3.15) *and* (3.16) *which guarantee the oscillation of the equation* (3.1) *in the cases* $\alpha = 1$ *and* $\alpha > 1$ *respectively. Meaning, we get,*

$$\sum\nolimits_{i=1}^{i=k} \int_t^{\tau(t)} q_i(s)\, ds > \psi_a(\alpha), \quad \text{for } t \geq t_1.$$

Remark. *If we consider a advanced differential equation on time scale of the form*

$$u^{\Delta}(t) - \sum\nolimits_{i=1}^{i=k} q_i(t)\, u^{\alpha}(\tau_i(t)) = 0, \quad \text{for } t \in [t_0, \infty) \tag{3.20}$$

on an arbitrary time scale $\mathbb{T}$ *with* $\sup \mathbb{T} = \infty$*. Thus, equation* (3.1) *becomes a special case of equation* (3.20) *in a case* $\mathbb{T} = \mathbb{R}$*. From the method given in this paper, one can obtain some oscillation criteria for* (3.20)*. It means obtaining generalizations of the Theorems 3.2.1 and 3.2.2. The details are left to the reader.*

Chapter 4

An improved oscillation result for advanced differential equations on time scale

In this chapter, we use the recursive sequence we have constructed to establish some new oscillation results of first-order linear dynamic equations with damping. The original results of this chapter are published in [48].

4.1 Introduction

In this chapter, we consider the advanced differential equation on time scale of the form

$$u^{\Delta}(t) - \eta(t)\, u(\lambda(t)) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{4.1}$$

on a time scale $\mathbb{T}$, since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$, with $t_0 \in \mathbb{T}$.
The equation (4.1) will be studied under the following assumptions:

($\mathcal{H}_1$) The function $\eta \in \mathcal{C}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$, such as $\eta \neq 0$ on any interval of the form $[t_0, \infty)_{\mathbb{T}}$.

($\mathcal{H}_2$) The function $\lambda \in \mathcal{C}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$, such as

$$\lambda(t) > t, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \quad \text{and} \quad \lim_{t\to\infty} \lambda(t) = \infty.$$

By a solution of (4.1) we mean a nontrivial real-valued function $u \in \mathcal{C}^1([T_u, \infty)_{\mathbb{T}}, \mathbb{R})$, $T_u \in [t_0, \infty)_{\mathbb{T}}$ which satisfies (4.1) on $[T_u, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (3.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (4.1) is called oscillatory if all its solutions are oscillatory. So far, there are any results on oscillatory of (4.1). Hence the aim of this chapter is to give some oscillation criteria for this equation.

4.2 Oscillation Results

To derive main oscillation in this section, we need the following lemma.

Definition 4.2.1. *Let us define a sequence of functions by the recurrence relation*

$$w_{n+1}(t) := \int_t^{\lambda(t)} \eta(s) \exp(w_n(t)) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.2}$$

with

$$w_0(t) := \int_t^{\lambda(t)} \eta(s) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.3}$$

Lemma 4.2.1. *If u is an positive solution of (4.1), then the sequence $\{w_n(t) : n \in \mathbb{N}\}$ converges.*

Proof.
Let u be an eventually positive solution of (3.1). From (3.1), we have $u^{\Delta}(t) \geq 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$, by Pötzsche's chain rule 1.1.5, we see that

$$\begin{aligned} (\ln(u(t)))^{\Delta} &= u^{\Delta}(t) \int_0^1 (hu(t) + (1-h) u^{\sigma}(t))^{-1} dh \\ &\geq \frac{u^{\Delta}(t)}{u(t)}, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \end{aligned}$$

so, we get

$$\begin{aligned} \ln\left(\frac{u(\lambda(t))}{u(t)}\right) &\geq \int_t^{\lambda(t)} \frac{u^{\Delta}(s)}{u(s)} \Delta s = \int_t^{\lambda(t)} \eta(s) \frac{u(\lambda(s))}{u(s)} \Delta s \\ &\geq \int_t^{\lambda(t)} \eta(s) \Delta s = w_0(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned} \tag{4.4}$$

This means,

$$\frac{u(\lambda(t))}{u(t)} \geq \exp(w_0(t)), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Multiplying the left-hand side by $\eta(t)$, we get

$$\frac{\eta(t)\, u(\lambda(t))}{u(t)} \geq \eta(t) \exp(w_0(t)), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

It follows from (4.4) and the above inequality, we obtain

$$\begin{aligned} \ln\left(\frac{u(\lambda(t))}{u(t)}\right) &\geq \int_t^{\lambda(t)} \eta(s) \exp(w_0(s))\, \Delta s \\ &= w_1(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

By induction, we can see that if

$$\ln\left(\frac{u(\lambda(t))}{u(x)}\right) \geq w_n(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

In the same way, we find that the inequality is true for $n+1$. By (4.2) and the above inequality, we conclude that the sequence $\{w_n(t) : n \in \mathbb{N}\}$ is increasing and increased, then $\{w_n(t) : n \in \mathbb{N}\}$ is converges. □

Lemma 4.2.2. *The sequence $\{w_n(t) : n \in \mathbb{N}\}$ defined by (4.2), converges if and only if*

$$\int_t^{\lambda(t)} \eta(s)\, \Delta s \leq \frac{1}{e}, \quad \textit{for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.5}$$

Proof.

Sufficient: Suppose that (4.3) is true. Then

$$w_0(t) \leq \frac{1}{e} = a_0, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}},$$

Then, we get

$$\begin{aligned} w_1(t) &\leq \int_t^{\lambda(t)} \eta(s) \exp(w_0(t))\, \Delta s \\ &\leq a_0 \exp(a_0) = a. \end{aligned}$$

By induction, we can see that if

$$w_n(t) \leq a_0 \exp(a_n) < 1.$$

In view of Lemma 1.2.4, $\{w_n(t) : n \in \mathbb{N}\}$ converges.

Necessary: Suppose that $\{w_n(t) : n \in \mathbb{N}\}$ converges. then there is a positive real function denoted $w(t)$, such as $w(t) = \lim_{n\to\infty} w_n(t)$, by (4.2), we find that the function w is satisfied

$$w(t) = \int_t^{\lambda(t)} \eta(s) \exp(w(s)) \Delta s, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

The above equality, we conclude that the function w is increased on $[t_0, \infty)_{\mathbb{T}}$. Let $\psi(t) = \exp(w(t)) \geq 1$, for $t \in [t_0, \infty)_{\mathbb{T}}$, we have

$$\psi(t) = \exp\left(\int_t^{\lambda(t)} \eta(s) \exp(w(s)) \Delta s\right), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

It follows from ψ is increased on $[t_0, \infty)_{\mathbb{T}}$ and the above equality, we obtain

$$\begin{aligned} \exp\left(\psi(t) \int_t^{\lambda(t)} \eta(s) \Delta s\right) &\leq \exp\left(\int_t^{\lambda(t)} \eta(s) \psi(s) \Delta s\right) \\ &= \psi(t), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned}$$

Then,

$$\int_t^{\lambda(t)} \eta(s) \Delta s \leq \frac{\ln(\psi(t))}{\psi(t)}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.6}$$

On the other hand, we have

$$\max\left\{\frac{\ln(x)}{x} : x \geq 1\right\} = \frac{1}{e}.$$

By (4.6) and the above inequality, we have

$$\int_t^{\lambda(t)} \eta(s) \Delta s \leq \frac{1}{e}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

This completes the proof. □

Remark. *If u is an positive solution of (4.1), then inequality (4.5) is satisfied.*

Now, we establish some sufficient conditions which guarantee that every solution u of (4.1) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

Theorem 4.2.1. *For all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such as*

$$\int_t^{\lambda(t)} \eta(s) \Delta s > \frac{1}{e}, \quad \textit{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{4.7}$$

Then any solution of (4.1) is oscillatory.

Proof.

Suppose that (4.1) has a nonoscillatory solution u on $[t_0, \infty)_{\mathbb{T}}$. Since $-u$ is also a solution of (4.1), we can confine our discussion only to the case where the solution u is eventually positive solution of (4.1). We may assume without loss of generality that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that

$$u\left(t\right) > 0 \quad \text{and} \quad u\left(\lambda\left(t\right)\right) > 0, \text{ for all } t \in [t_1, \infty)_{\mathbb{T}}.$$

This means the following equation (4.1) has a positive solution u on $[t_1, \infty)_{\mathbb{T}}$.

$$u^{\Delta}\left(t\right) - \eta\left(t\right) u\left(\lambda\left(t\right)\right) = 0, \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}$$

By Lemma 4.2.1 and Lemma 4.2.2, we obtain

$$\int_{t}^{\lambda(t)} \eta\left(s\right) \Delta s \leq \frac{1}{e}, \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}.$$

which contradicts (4.7).

This completes the proof. □

As a Theorem of the previous result, we deduce the following corollaries.

Corollary 4.2.1. *If*

$$\liminf_{t \to \infty} \int_{t}^{\lambda(t)} \eta\left(s\right) \Delta s > \frac{1}{e}$$

Then any solution of (4.1) is oscillatory.

Corollary 4.2.2. *If*

$$\limsup_{t \to \infty} \int_{t}^{\lambda(t)} \eta\left(s\right) \Delta s > 1$$

Then any solution of (4.1) is oscillatory.

4.3 Application

In this section, we give applications and examples where we apply Theorem 4.2.1 to some particular cases.

Next, we consider the advanced differential equation on time scale of the form

$$u^{\Delta}\left(t\right) + q\left(t\right) u^{\sigma}\left(t\right) - \eta\left(t\right) u\left(\lambda\left(t\right)\right) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.8}$$

and

$$u^{\Delta}\left(t\right) - q\left(t\right) u\left(t\right) - \eta\left(t\right) u\left(\lambda\left(t\right)\right) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.9}$$

with, the functions $q \in \mathcal{C}\left([t_0, \infty)_{\mathbb{T}}, [0, \infty)\right)$.

Theorem 4.3.1. *For all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such as*

$$\int_t^{\lambda(t)} \eta(s)\, e_{\ominus q}(s, t_0)\, e_{\ominus q}(\tau(s), t_0)\, \Delta s > \frac{1}{e}, \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.10}$$

Then any solution of (4.8) is oscillatory.

Proof.

By equation (4.8), we find

$$[u(t)\, e_q(t, t_0)]^{\Delta} = \eta(t)\, e_{\ominus q}(t, t_0)\, u(\lambda(t)), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Let $v(t) = u(t)\, e_q(t, t_0)$, for $t \in [t_0, \infty)_{\mathbb{T}}$, we have

$$v^{\Delta}(t) = \eta(t)\, e_{\ominus q}(t, t_0)\, e_{\ominus q}(\tau(t), t_0)\, v(\lambda(t)), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

we conclude that the latter's equation is the same as the equation (4.1). And from it we conclude if it is achieved (4.10), then any solution of (4.8) is oscillatory. □

Theorem 4.3.2. *For all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such as*

$$\int_t^{\lambda(t)} \eta(s)\, \frac{e_q(\lambda(s), t_0)}{e_q^{\sigma}(s, t_0)} \Delta s > \frac{1}{e}, \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.11}$$

Then any solution of (4.9) is oscillatory.

Proof.

Let u be an eventually positive solution of (4.9), then

$$\left[\frac{u(t)}{e_q(t, t_0)}\right]^{\Delta} = \frac{\eta(t)}{e_q(t, t_0)\, e_q^{\sigma}(t, t_0)} u(\lambda(t)), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Let

$$v(t) = \frac{u(t)}{e_q(t, t_0)}, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

we have

$$v^{\Delta}(t) = \eta(t)\, \frac{e_q(\lambda(t), t_0)}{e_q(t, t_0)} u(\lambda(t)), \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

we conclude that the latter's equation is the same as the equation (4.1). And from it we conclude if it is achieved (4.11), then any solution of (4.9) is oscillatory. □

Example 8. *Consider the delay differential equation*

$$x^{\Delta}(t) - (t+1)\, x(t+1) = 0, \quad \text{for all } t \in \mathbb{N}. \tag{4.12}$$

Here,

$$\mathbb{T} = \mathbb{N}, \eta(t) = t+1, \quad \text{and } \lambda(t) = t+1 > t, \quad \text{for all } t \in \mathbb{N}.$$

On the other hand, we have

$$\int_t^{\lambda(t)} \eta(s)\,\Delta s = \int_t^{t+1} (s+1)\,\Delta s = t+1 > \frac{1}{e}, \quad \text{for all } t \in \mathbb{N}.$$

Thus, (4.7) holds. By Theorem 4.2.1, equation (4.12) is oscillatory.

Example 9. *Consider the delay differential equation*

$$x^{\Delta}(t) - tx(2t) = 0, \quad \text{for all } t \in \overline{2^{\mathbb{N}}}. \tag{4.13}$$

Here,

$$\mathbb{T} = \overline{2^{\mathbb{N}}}, \quad \eta(t) = t, \quad \text{and} \quad \lambda(t) = 2t > t, \quad \text{for all } t \in \overline{2^{\mathbb{N}}}.$$

Then

$$\int_t^{\lambda(t)} \eta(s)\,\Delta s = \int_t^{2t} s\Delta s = 2t^2 \geq 1 = \lambda, \text{ for all } t \in \overline{2^{\mathbb{N}}}.$$

Thus, (4.7) holds. By Theorem 4.2.1, equation (4.13) is oscillat

Conclusion and future perspectives

In this thesis, we study the oscillation of some differential equations. The equation we started with is of the type of fourth order hybird nonlinear functional dynamic equations with damping to find the oscillation of this equation, we used the following methods:

▷ The generalized Riccati transformation technique.

▷ The integral averaging technique.

We have also studied the oscillation of the semi-linear equation and the linear equations on time scalle by using the following method new technique "**Recursive Sequence**".

For future researches, we can look for oscillation of solution for:

▷ We study a class of higher order hybird advanced differential systems, for example

$$\left(\frac{a\left(t\right)\left(u^{(k-2)}\left(t\right)\right)^{\beta}}{f\left(t,u\left(t\right)\right)}\right)^{(2)}+\sum_{i=1}^{i=n}b_{i}\left(t\right)u^{\beta}\left(\tau_{i}\left(t\right)\right)=g\left(t,u\left(\eta\left(t\right)\right)\right), \quad \text{for all}\, t\geq t_{0},$$

what is considered to extend the results of the article [3].

▷ We use the recursive sequence we have constructed establish some new oscillation results of first-order non-linear dynamic equations with damping, , for example

$$u^{\Delta}\left(t\right)-\sum_{i=1}^{i=n}q_{i}\left(t\right)f_{i}\left(u^{\alpha}\left(\tau\left(t\right)\right)\right)=0, \quad \text{for } t\geq t_{0},$$

what is considered to extend the results of the article [4, 48].

▷ We study oscillation of solution for certain fractional partial differential equations, for example,

$$\frac{\partial}{\partial t}\left(\mathcal{D}_{+,t}^{\alpha}u\left(x,t\right)\right)=a\left(t\right)\triangle u\left(x,t\right)-m\left(x,t,u\left(x,t\right)\right)+f\left(x,t\right), \quad \text{for all } \left(x,t\right)\in\Omega\times\mathbb{R}_{+},$$

where $\mathcal{D}_{+,t}^{\alpha}u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t is defined by :

$$\mathcal{D}_{+,t}^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left(\int_0^t (t-s)^{-\alpha}u(x,s)\,ds\right), \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+,$$

where Γ is the gamma function.

With one of the two following boundary condition

$$\frac{\partial u(x,t)}{\partial N} = \psi(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

or

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\mathbb{R}_+ = [0,\infty)$, $\alpha \in (0,1)$ is constant, $\triangle$ is the Laplacian in $\mathbb{R}^n$, N is the unit exterior normal vector to $\partial\Omega$ and ψ is a continuous function on $\partial\Omega \times \mathbb{R}_+$.

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Abstract:

Our work will be devoted to the study of the properties of the oscillation and the asymptotic behaviour solutions of certain classes of differential equations.

We will establish a new criterion to check if all the solutions of a differential equation are oscillatory,non-oscillatory or asymptotic. We will determinate and prove this criterion is using a Generalized Riccati method as well as other methods.

Key words: Oscillation, Advanced differential equations, Hybird Differential Equation, Riccati Technique, Time scale.

AMS Classifications : 34K06, 34K11, 34C10, 34K11.

Résumé:

Nos travaux seront consacrés à l'éude des propriétés de l'oscillation et le comportement asymptotique des solutions des certaines classes équations différentielles.

Nous établirons un nouveau critère pour vérifier si toutes les solutions d'une équation différentielles est oscillatoire, non oscillatoires ou asymptotique. Nous prouvons ce critère en utilisant une technique de Riccati Généralisé et d'autres méthodes.

Mots et Phrases Clefs:

Oscillation, Équations différentielles avancées, Équation différentielle Hybird, Technique de Riccati, Échelle de temps.

Classification AMS: 34K06, 34K11, 34C10, 34K11.