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THE PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
UNIVERSITY OF SCIENCE AND TECHNOLOGY OF
ORAN MOHAMED BOUDIAF
Faculty of Mathematics and Computer Science
Department of Mathematics



Handout of Algebra 02
Courses and Exercises for Algebra 02

Author: Dr. ANBER Ahmed

Academic year: 2025-2026

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INTRODUCTION

This handout was intended for students enrolled in the first year of the LMD system, computer science, first semester of the academic year.

The content of this handout corresponds to the official program for the subject Algebra 2 taught in the first year.

The manuscript contains four chapters :

- Vector spaces
- Linear applications
- Matrices
- Systems of linear equations

At the end of each chapter, solved exercises and equivalent terminologies in French are given.

Vector spaces

In this chapter, $\mathbb{k}$ denotes a commutative field.

1.1 Definition of a vector space

Definition 1.1

We call vector space on $\mathbb{k}$ or $\mathbb{k}$ -vector space, any not empty set E equipped with two laws :

- an internal composition law (noted $+$)

$$+ : E \times E \rightarrow E$$

$$(u, v) \mapsto u + v$$

- an external composition law (noted $\bullet$)

$$\bullet : \mathbb{k} \times E \rightarrow E$$

$$(\lambda, u) \mapsto \lambda \bullet u$$

which check the following properties :

1. $\forall u, v \in E, u + v = v + u$ (*commutative*)

2. $\forall u, v, w \in E, u + (v + w) = (u + v) + w$ (*associative*)

3. $+$ has a neutral element : $\exists 0_E \in E, \forall u \in E, u + 0_E = u$.

4. All $u \in E$ admits a symmetrical $u' : \forall u \in E, \exists u' \in E, u + u' = 0_E$.

$$5. \forall u \in E, 1_{\mathbb{k}}.u = u.$$

$$6. \forall u \in E, \forall \alpha, \beta \in \mathbb{k}, \alpha \bullet (\beta \bullet u) = (\alpha\beta) \bullet u .$$

$$7. \forall u \in E, \forall \alpha, \beta \in \mathbb{k}, (\alpha + \beta) \bullet u = \alpha \bullet u + \beta \bullet u.$$

$$8. \forall u, v \in E, \forall \alpha \in \mathbb{k}, \alpha \bullet (u + v) = \alpha \bullet u + \alpha \bullet v .$$

Definition 1.2

The elements of a vector space are called vectors. Those of the field $\mathbb{k}$ are called scalars.

Example 1.1

Let $\mathbb{k} = \mathbb{R}$ and $E = \mathbb{R}^2$. We define a law of internal composition as follows

$$\begin{aligned} + : E \times E &\rightarrow E \\ ((x, y), (x', y')) &\mapsto (x, y) + (x', y') = (x + x', y + y') \end{aligned}$$

and the external composition law by

$$\begin{aligned} \bullet : \mathbb{k} \times E &\rightarrow E \\ (\lambda, (x, y)) &\mapsto \lambda \bullet (x, y) = (\lambda x, \lambda y) \end{aligned}$$

The neutral element of the internal law is $0_{\mathbb{R}^2} = (0, 0)$ and the symmetric of (x, y) is $(-x, -y)$.

Example 1.2

The following sets are $\mathbb{R}$ -vector spaces : $\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}_n[X]$.

Example 1.3

Let $E = \mathbb{R}^2$ equipped with two laws :

- an internal composition law (noted +)

$$\begin{aligned} + : E \times E &\rightarrow E \\ ((x, y), (x', y')) &\mapsto (x + x', y + y') \end{aligned}$$

- an external composition law (noted $\bullet$)

$$\begin{aligned} \bullet : \mathbb{k} \times E &\rightarrow E \\ (\lambda, (x, y)) &\mapsto (\lambda x, 0) \end{aligned}$$

E is it a $\mathbb{R}$ -vector space?

For $u = (x, y) \in E$, we have

$$1_{\mathbb{K}} \bullet u = (1 \times x, 0) \stackrel{?}{=} (x, y).$$

Then, there exists $u = (1, 2) \in E$, such that $1_{\mathbb{K}} \bullet u \neq u$, so, E is not a $\mathbb{R}$ -vector space.

Proposition 1.1

Let E be a $\mathbb{K}$ -vector space. We have the following properties :

1. $\forall u \in E, 0 \bullet u = 0_E$.
2. $\forall \lambda \in \mathbb{K}, \lambda \bullet 0_E = 0_E$.
3. $\forall u \in E, (-1) \bullet u = -u$.
4. $\forall u \in E, \forall \lambda \in \mathbb{K}, \lambda \bullet u = 0_E \Leftrightarrow \lambda = 0 \text{ or } u = 0_E$.

Proof

1. Let $u \in E$, we have

$$u + 0 \bullet u = 1 \bullet u + 0 \bullet u = (1 + 0) \bullet u = 1 \bullet u = u$$

and

$$0 \bullet u + u = 0 \bullet u + 1 \bullet u = (0 + 1) \bullet u = 1 \bullet u = u$$

Hence, $0 \bullet u$ is a neutral element for $+$ and by uniqueness, we have $0 \bullet u = 0_E$.

2. let $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} 0_E + 0_E &= 0_E \Leftrightarrow \lambda \bullet (0_E + 0_E) = \lambda \bullet 0_E \\ &\Leftrightarrow \lambda \bullet 0_E + \lambda \bullet 0_E = \lambda \bullet 0_E \\ &\Leftrightarrow \lambda \bullet 0_E = 0_E. \end{aligned}$$

3. Let $u \in E$, we have

$$u + (-1) \bullet u = 1 \bullet u + (-1) \bullet u = (1 + (-1)) \bullet u = 0 \bullet u = 0_E$$

and

$$(-1) \bullet u + u = (-1) \bullet u + 1 \bullet u = ((-1) + 1) \bullet u = 0 \bullet u = 0_E$$

Thus, $(-1) \bullet u = -u$.

4.

($\Leftarrow$) From (1) and (2) we deduce that if $(\lambda = 0 \text{ or } u = 0_E)$, then $\lambda \bullet u = 0_E$.

($\Rightarrow$) Let $u \in E$ and $\lambda \in \mathbb{k}$, suppose that $\lambda \bullet u = 0_E$. If $\lambda \neq 0$, we have

$$u = 1 \bullet u = \left(\frac{1}{\lambda}\lambda\right) \bullet u = \left(\frac{1}{\lambda}\right) \bullet (\lambda \bullet u) = \left(\frac{1}{\lambda}\right) \bullet 0_E = 0_E.$$

1.2 Vector subspace

Definition 1.3

A subset F of a $\mathbb{k}$ -vector space E is called a vector subspace if it is non-empty and itself forms a $\mathbb{k}$ -vector space with the operations restricted to F : internal addition $(+)$ and external scalar multiplication $(\bullet)$.

Proposition 1.2

A subset F of $\mathbb{k}$ -vector space E is a vector subspace if and only if :

1. $F \neq \emptyset$ (F contains the zero element 0_E).
2. For all $u, v \in F$, $u + v \in F$
3. For all $\lambda \in \mathbb{k}$ and all $u \in F$, $\lambda \bullet u \in F$.

Proposition 1.3

A subset F of $\mathbb{k}$ -vector space E is a vector subspace if and only if :

1. $F \neq \emptyset$ (F contains the zero element 0_E).
2. For all $\alpha, \beta \in \mathbb{k}$ and all $u, v \in F$, $\alpha \bullet u + \beta \bullet v \in F$

Example 1.4

Let F_1, F_2 and F_3 be three subset of $\mathbb{R}^3$, such that

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 / x + y - 5z = 0\}$$

$$F_2 = \{(x, y, z) \in \mathbb{R}^3 / xy - z = 0\}$$

$$F_3 = \{(x, y, z) \in \mathbb{R}^3 / x + z + 1 = 0\}$$

- F_1 is-it a vector subspace of $\mathbb{R}^3$?

We have :

(i). F_1 contains the zero vector, as $0_{\mathbb{R}^3} = (0, 0, 0) \in F_1$.

(ii). Let $u = (x, y, z), v = (x', y', z') \in F_1, u + v = (x + x', y + y', z + z') \in F_1$?

$$\begin{aligned} (x + x') + (y + y') - 5(z + z') &= (x + y - 5z) + (x' + y' - 5z') \\ &= (0) + (0) = 0, \end{aligned}$$

so, $u + v \in F_1$.

(iii). Let $u = (x, y, z) \in F_1, \lambda \in \mathbb{k} = \mathbb{R}, \lambda u = (\lambda x, \lambda y, \lambda z) \in F_1$?

$$\begin{aligned} (\lambda x) + (\lambda y) - 5(\lambda z) &= \lambda(x + y - 5z) \\ &= \lambda(0) = 0, \end{aligned}$$

so, $\lambda u \in F_1$

Thus, F_1 is a subspace.

- F_2 is not a vector subspace of $\mathbb{R}^3$, because for $u = (1, 1, 1), v = (-2, -2, 4)$, we have

$$u, v \in F_2 \quad \text{but} \quad u + v = (-1, -1, 5) \notin F_2.$$

- F_3 is not a vector subspace of $\mathbb{R}^3$, because $0_{\mathbb{R}^3} = (0, 0, 0) \notin F_3$.

Proposition 1.4

Let E be a $\mathbb{k}$ -vector space and $(F_i)_{i \in I}$ a family of vector subspaces of E . Then, $\bigcap_{i \in I} F_i$ is a vector subspace of E .

Proof

Let us note $F = \bigcap_{i \in I} F_i$, we have

(i). F contains the zero vector, because $0_E \in F_i$.

(ii). Let $u, v \in F, u + v \in F$?

$$u, v \in F \Leftrightarrow u, v \in \bigcap_{i \in I} F_i \Leftrightarrow u, v \in F_i$$

$$\Leftrightarrow u + v \in F_i$$

$$\Leftrightarrow u + v \in \bigcap_{i \in I} F_i$$

so, $u + v \in F$.

(iii). Let $u \in F$, $\lambda \in \mathbb{k}$, $\lambda u \in F$?

$$u \in F \Leftrightarrow u \in \bigcap_{i \in I} F_i \Leftrightarrow u \in F_i$$

$$\Leftrightarrow \lambda u \in F_i$$

$$\Leftrightarrow \lambda u \in \bigcap_{i \in I} F_i$$

so, $\lambda u \in F$

Thus, F is a subspace.

Remark

The union of vector subspaces is not generally a vector subspace.

Indeed, for

$$F_1 = \{(x, y) \in \mathbb{R}^2 / x = 0\} \text{ and } F_2 = \{(x, y) \in \mathbb{R}^2 / y = 0\}.$$

We have : F_1 and F_2 are vector subspaces of $\mathbb{R}^2$ but $F_1 \cup F_2$ is not a vector subspace of $\mathbb{R}^2$, because for

$$u = (0, 1) \text{ and } v = (1, 0),$$

we have :

$$u, v \in F_1 \cup F_2 \text{ but } u + v = (1, 1) \notin F_1 \cup F_2.$$

1.3 Linear combination

Definition 1.4

Let $u_1, u_2, \dots, u_n$ be n vectors in a $\mathbb{k}$ -vector space E . Any vector of the form $u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$, where $\alpha_i \in \mathbb{k}$, is called a linear combination of the vectors $u_1, u_2, \dots, u_n$.

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called coefficients of the linear combination.

- We denote by $Vect(u_1, u_2, \dots, u_n)$ (or $\text{Span}(u_1, u_2, \dots, u_n)$) the set of all linear combinations of the vectors $u_1, u_2, \dots, u_n$.

$$X \in Vect(u_1, u_2, \dots, u_n) \Leftrightarrow \exists \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{k}, X = \sum_{i=1}^n \alpha_i u_i.$$

Example 1.5

In the $\mathbb{R}$ -vector space $\mathbb{R}^3$, $X = (0, 3, 5)$ is a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (2, -1, 0)$ and $u_3 = (0, 0, 1)$, because there exists $(\alpha_1, \alpha_2, \alpha_3) = (2, -1, 3) \in \mathbb{R}^3$, such that

$$2u_1 - u_2 + 3u_3 = X.$$

Remark

Let X_1, X_2 be two vectors in a $\mathbb{k}$ -vector space E , such that $X_1 = \lambda X_2$, with $\lambda \in \mathbb{k}$. Then, we say that X_1 and X_2 are collinear.

Proposition 1.5

Let $u_1, u_2, \dots, u_n$ be n vectors in a $\mathbb{k}$ -vector space E . The set $Vect(u_1, u_2, \dots, u_n)$ is a vector subspace of E .

Proof

(i). We have

$$0_E = \sum_{i=1}^n (0 \times u_i),$$

then $Vect(u_1, u_2, \dots, u_n)$ contains the zero vector.

(ii). Let $X, Y \in Vect(u_1, u_2, \dots, u_n)$, and $\alpha, \beta \in \mathbb{k}$, $\alpha X + \beta Y \in Vect(u_1, u_2, \dots, u_n)$?

$$X, Y \in Vect(u_1, u_2, \dots, u_n) \Leftrightarrow X = \sum_{i=1}^n \alpha_i u_i \text{ and } Y = \sum_{i=1}^n \beta_i u_i, \text{ with } \alpha_i, \beta_i \in \mathbb{k}$$

$$\Rightarrow \alpha X + \beta Y = \alpha \sum_{i=1}^n \alpha_i u_i + \beta \sum_{i=1}^n \beta_i u_i = \sum_{i=1}^n \underbrace{(\alpha \alpha_i + \beta \beta_i)}_{\lambda_i} u_i \in Vect(u_1, u_2, \dots, u_n)$$

Thus, $Vect(u_1, u_2, \dots, u_n)$ is a subspace.

Remark

For $\lambda_i \in \mathbb{k}$, we have

$$Vect(u_1, u_2, \dots, u_n, \sum_{i=1}^n \lambda_i u_i) = Vect(u_1, u_2, \dots, u_n).$$

1.4 Direct Sums

Definition 1.5

Let E be a $\mathbb{k}$ -vector space and F_1, F_2 be two vector subspaces of E . We define the set $F_1 + F_2$

as follows

$$F_1 + F_2 = \{u + v \mid u \in F_1 \text{ and } v \in F_2\}.$$

Example 1.6

Let F_1, F_2 be two vector subspaces of $\mathbb{R}^3$, defined by

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 0\}$$

$$F_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x - 2y = 0\}$$

We have the vector $u = (1, 2, 3) \in F_1 + F_2$, because there exists two vectors $X_1 = (1, 2, 4) \in F_1$ and $X_2 = (0, 0, -1) \in F_2$ such that $u = X_1 + X_2$.

Is the vector $w = (1, 2, 3) \in F_1 + F_2$?

Can we find two vectors $X = (x, y, z) \in F_1$ and $Y = (x', y', z') \in F_2$ such that $w = X + Y$.

We solve the vector equation $w = X + Y$

$$w = X + Y \Leftrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Leftrightarrow \begin{cases} x + x' = 1 \\ y + y' = 2 \\ z + z' = 3 \end{cases},$$

and moreover we have $2x + y - z = 0$ and $x' - 2y' = 0$. Then we obtain the following system

$$\begin{cases} x + x' = 1 \\ y + y' = 2 \\ z + z' = 3 \\ 2x + y - z = 0 \\ x' - 2y' = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y' = 1 \\ y + y' = 2 \\ 2x + y + z' = 3 \\ z = 2x + y \\ x' = 2y' \end{cases} \Leftrightarrow \begin{cases} x = 1 - 2y' \\ y = 2 - y' \\ z' = -1 + 5y' \\ z = 4 - 5y' \\ x' = 2y' \end{cases}$$

Then,

$$X = (1 - 2y', 2 - y', 4 - 5y') \quad \text{and} \quad Y = (2y', y', -1 + 5y').$$

Proposition 1.6

The finite sum of subspaces of a $\mathbb{k}$ -vector space E is a subspace of E .

Proof

Let $F_1, F_2, \dots, F_n$ be n vector subspaces of $\mathbb{k}$ -vector space E .

Let us note $F = \sum_{i=1}^n F_i$, we have

(i). F contains the zero vector, because $\underbrace{0_E + 0_E + \dots + 0_E}_{n \text{ times}} = 0_E \in F$.

(ii). Let $u, v \in F$, and $\alpha, \beta \in \mathbb{k}$, $\alpha u + \beta v \in F$?

$$\begin{aligned} u, v \in F &\Leftrightarrow u, v \in \sum_{i=1}^n F_i \Leftrightarrow u = \sum_{i=1}^n u_i, v = \sum_{i=1}^n v_i, \text{ with } u_i, v_i \in F_i \\ &\Rightarrow \alpha u + \beta v = \alpha \sum_{i=1}^n u_i + \beta \sum_{i=1}^n v_i = \sum_{i=1}^n (\alpha u_i + \beta v_i) \in \sum_{i=1}^n F_i \\ &\Rightarrow \alpha u + \beta v \in F. \end{aligned}$$

Thus, F is a subspace.

Proposition 1.7

Let F_1, F_2 be two vector subspaces of a $\mathbb{k}$ -vector space E and B_1 a generating family of F_1 and B_2 a generating family of F_2 . Then $B_1 \cup B_2$ is a generating family of $F_1 + F_2$.

$$F_1 + F_2 = Vect(B_1 \cup B_2) \quad \text{or} \quad F_1 + F_2 = Span(B_1 \cup B_2).$$

Definition 1.6

Let F_1, F_2 be two vector subspaces of a $\mathbb{k}$ -vector space E . We say that the sum $F_1 + F_2$ is direct (or that F_1 and F_2 are in direct sum) and we denote $F_1 \oplus F_2$, if any vector X of $F_1 + F_2$ can be uniquely expressed as $X = u + v$, with $u \in F_1$ and $v \in F_2$.

Proposition 1.8

Let F_1, F_2 be two vector subspaces of a $\mathbb{k}$ -vector space E . The sum $F_1 + F_2$ is direct if and only if $F_1 \cap F_2 = \{0_E\}$.

Definition 1.7 (Supplementary Subspaces)

Let F_1, F_2 be two vector subspaces of a $\mathbb{k}$ -vector space E . We say that F_1 and F_2 are supplementary in E if $F_1 \oplus F_2 = E$. (That is, $F_1 \cap F_2 = \{0_E\}$ and $F_1 + F_2 = E$).

In other words, F_1 and F_2 are supplementary in E if every vector X in E can be uniquely expressed as $X = u + v$ with $u \in F_1$ and $v \in F_2$.

Example 1.7

Let F_1, F_2 be two vector subspaces of $\mathbb{R}^3$, defined by

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 / x - 3y + 2z = 0\}$$

$$F_2 = \{(x, y, z) \in \mathbb{R}^3 / y = z = 0\}$$

We have

$$(i). F_1 \cap F_2 = \{0_{\mathbb{R}^3}\}$$

Let $X = (x, y, z) \in \mathbb{R}^3$,

$$X \in F_1 \cap F_2 \Leftrightarrow X \in F_1 \text{ and } X \in F_2,$$

we solve the equation $X_{F_1} = X_{F_2}$.

We have :

$$X \in F_1 \Leftrightarrow x - 3y + 2z = 0$$

and

$$X \in F_2 \Leftrightarrow y = z = 0$$

So, we obtain the following system :

$$\begin{cases} x - 3y + 2z = 0 \\ y = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 3y - 2z = 0 \\ y = 0 \\ z = 0 \end{cases}$$

then,

$$F_1 \cap F_2 = \{0_{\mathbb{R}^3}\}.$$

$$(ii). F_1 + F_2 = \mathbb{R}^3$$

We must show that any vector $u = (x, y, z)$ of $\mathbb{R}^3$ can be written in the form $u = X_{F_1} + X_{F_2}$, such that $X_{F_1} = (x_1, y_1, z_1) \in F_1$ and $X_{F_2} = (x_2, y_2, z_2) \in F_2$.

We solve the vector equation $u = X_{F_1} + X_{F_2}$

$$u = X_{F_1} + X_{F_2} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 + x_2 = x \\ y_1 + y_2 = y \\ z_1 + z_2 = z \end{cases},$$

and moreover we have $x_1 - 3y_1 + 2z_1 = 0$ and $y_2 = z_2 = 0$. Then we obtain the following system

$$\begin{cases} x_1 + x_2 = x \\ y_1 + y_2 = y \\ z_1 + z_2 = z \\ x_1 - 3y_1 + 2z_1 = 0 \\ y_2 = z_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = x - 3y + 2z \\ y_1 = y \\ z_1 = z \\ x_1 = 3y - 2z \\ y_2 = z_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 - 2y' \\ y = 2 - y' \\ z' = -1 + 5y' \\ z = 4 - 5y' \\ x' = 2y' \end{cases}$$

Then,

$$X_{F_1} = (3y - 2z, y, z) \text{ and } X_{F_2} = (x - 3y + 2z, 0, 0).$$

From (i) and (ii) we deduce that : $F_1 \oplus F_2 = \mathbb{R}^3$.

1.5 Linearly independent, dependent and generating family, bases

1.5.1 Linearly independent, dependents

Definition 1.8

Let E be a $\mathbb{k}$ -vector space and $B = (u_1, u_2, \dots, u_n)$ a family of n vectors of E .

- The family B is said to be linearly independent if and only if

$$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{k},$$

$$\sum_{i=1}^n \alpha_i u_i = 0_E \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$

- Otherwise, we say that the family is linearly dependent. That is to say there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zeros, in $\mathbb{k}$ satisfying $\sum_{i=1}^n \alpha_i u_i = 0_E$.

$$\exists (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{k}^n - \{(0, 0, \dots, 0)\},$$

$$\sum_{i=1}^n \alpha_i u_i = 0_E.$$

Example 1.8

Let B_1 and B_2 be two sets of vectors in $\mathbb{R}^3$ defined by :

$$\begin{aligned} B_1 &= \{v_1 = (1, 2, -1), v_2 = (-3, -6, 3)\} \\ B_2 &= \{u_1 = (1, 2, 0), u_2 = (1, 1, 1), u_3 = (0, 1, 0)\}. \end{aligned}$$

The family B_1 is linearly dependent in $\mathbb{R}^3$, since $v_2 = -3v_1$. Geometrically, v_1 and v_2 lie on the same line. We say that the two vectors are collinear.

The family B_2 is linearly independent in $\mathbb{R}^3$, since for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$\begin{aligned} \sum_{i=1}^3 \alpha_i u_i = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_2 = 0 \end{cases} \\ &\Rightarrow \begin{cases} \alpha_1 = -\alpha_2 = 0 \\ \alpha_3 = -2\alpha_1 - \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \\ &\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \end{aligned}$$

1.5.2 Generating family

Definition 1.9

Let E be a $\mathbb{k}$ -vector space and $B = (u_1, u_2, \dots, u_n)$ a family of n vectors of E . B is said generating family or span of E if $E = \text{span}(B)$. In other words, any vector of E can be written as a linear combination of vectors of B .

Example 1.9

The family $B = \{v_1 = (1, 2), v_2 = (1, 0)\}$ is generates $\mathbb{R}^2$, because for all vector $X = (x, y)$ of $\mathbb{R}^2$, we can find two scalars $\alpha, \beta \in \mathbb{R}$, such that $X = \alpha v_1 + \beta v_2$.

Indeed, for $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} X = \alpha v_1 + \beta v_2 &\Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \alpha + \beta = x \\ 2\alpha = y \end{cases} \\ &\Leftrightarrow \begin{cases} \beta = x - \alpha = x - \frac{1}{2}y \\ \alpha = \frac{1}{2}y \end{cases} \end{aligned}$$

1.5.3 Bases

Definition 1.9

Let E be a $\mathbb{k}$ -vector space and $B = (u_1, u_2, \dots, u_n)$ a family of n vectors of E . B is said basis of E if any vector of E can be written uniquely as linear combination of vectors of B .

In other words,

$$\forall u \in E, \exists! (\alpha_1, \dots, \alpha_n) \in \mathbb{k}^n, \text{ such that } u = \sum_{i=1}^n \alpha_i u_i.$$

Proposition 1.9

Let E be a $\mathbb{k}$ -vector space and $B = (u_1, u_2, \dots, u_n)$ a family of n vectors of E . We say that B is a basis for E if and only if, it is linearly independent and generating family of E .

Example 1.10

a. Let be E a vector subspaces of $\mathbb{R}^3$, defined by

$$E = \{(x, y, z) \in \mathbb{R}^3 / x - y + 2z = 0\}.$$

Let $X = (x, y, z) \in E$, then $x - y + 2z = 0$, so, $x = y - 2z$.

Which gives,

$$X = (x, y, z) = (y - 2z, y, z) = (y, y, 0) + (-2z, 0, z) = y(1, 1, 0) + z(-2, 0, 1),$$

Therefore, E is the vector subspace of $\mathbb{R}^3$ spanned by the vectors $u_1 = (1, 1, 0)$ and $u_2 = (-2, 0, 1)$, since the family $B = \{u_1 = (1, 1, 0), u_2 = (-2, 0, 1)\}$ is linearly independent, because the vectors are not collinear, so it forms a basis for E .

b. The family $B = \{u_1 = (1, 1, 0), u_2 = (-2, 0, 1)\}$ does not form a basis for $\mathbb{R}^3$, because two vectors cannot generate $\mathbb{R}^3$.

1.6 Finite-Dimensional Vector Space

1.6.1 Dimension of a vector space

Definition 1.10

We say that a vector space E is of dimension *fini* if it has a generating family *fini* (A generating family having a number *fini* of vectors). Otherwise, we say that it is of dimension *infini*.

Theorem 1.1

Let E be a vector space of dimension *fini*. All bases of E have the same cardinality n .

Definition 1.11

This cardinality n is the dimension of the vector space E over the field $\mathbb{k}$, we denote by $\dim E$.

By convention, if $E = \{0_E\}$, we denote by $\dim E = 0$.

If F is a vector subspace of a vector space E , then $\dim F \leq \dim E$.

Proposition 1.10

Let E be a vector space of dimension n .

- Every free family of n vectors in E is a basis of E .
- Every generating family of n vectors in E is a basis of E .

This proposition allows us to prove that a family is a basis without having to verify both axioms (free and generating).

Proposition 1.11

Let E be a vector space of dimension finite n .

- Any free family of E has at most n elements.

- Any generating family of E has at least n elements.

Theorem 1.2 (Incomplete Basis Theorem)

Let E be a vector space of dimension n and $B = \{u_1, u_2, \dots, u_p\}$ a free family with $p < n$.

We can complete the family B with $n - p$ vectors from E to form a basis of E .

Theorem 1.3

Let E be a vector space of dimension finite and F a vector subspace of E .

Then F is of dimension finite and $\dim F \leq \dim E$. and furthermore

$$E = F \iff \dim F = \dim E.$$

Theorem 1.4

Let F_1, F_2 be two vector subspaces of dimension finite of a vector space E .

Then $F_1 + F_2$ and $F_1 \cap F_2$ are of dimension finite and

$$\dim(F_1 + F_2) = \dim(F_1) + \dim(F_2) - \dim(F_1 \cap F_2).$$

Proposition 1.12

Let F_1, F_2 be two vector subspaces of a vector space E of dimension finite. We have :

$$F_1 \oplus F_2 = E \iff \begin{cases} \dim(F_1) + \dim(F_2) = \dim(E) \\ F_1 \cap F_2 = \{0_E\} \end{cases}$$

Coordinates of a vector in a basis

Let E be a vector space, $F = \{u_1, u_2, \dots, u_n\}$ a basis of E , and u a vector of E :

This vector is uniquely written in the form

$$u = \sum_{i=1}^n \alpha_i u_i$$

The numbers α_i are called the coordinates of the vector u in the basis F .

1.7 Rank

1.7.1 Rank of a finite family of vectors

Definition 1.12

Let E be a finite vector space and $F = \{u_1, u_2, \dots, u_n\}$ a finite family of vectors in E .

We call the rank of this family and denote by $rk(F)$ the dimension of $Vect(u_1, u_2, \dots, u_n)$.

Proposition 1.13

1. The family F is free if and only if $rk(F) = n$:
2. The family F generates E if and only if $rk(F) = \dim E$.

Example 1.11

Let $\mathcal{F} = \{u_1 = (1, 1, 2), u_2 = (-1, 3, 5), u_3 = (0, 4, 7)\}$ be a family of three vectors of $\mathbb{R}^3$.

We consider F a vector subspace engendered by the vectors $u_1 = (1, 1, 2), u_2 = (-1, 3, 5)$ and $u_3 = (0, 4, 7)$

$$F = Span(\mathcal{F}).$$

The vectors are linearly dependent, because

$$u_3 = u_1 + u_2,$$

so,

$$Span(u_1, u_2, u_3) = Span(u_1, u_2),$$

and hence, u_1 and u_2 are not colinear, thus the family $\{u_1, u_2\}$ forms a basis of F . Then,

$$rk(\mathcal{F}) = \dim F = Card(\{u_1, u_2\}) = 2.$$

Proof

Let $F = Vect(u_1, u_2, \dots, u_n)$, then $rk(F) = \dim F$.

1. If the family $(u_1, u_2, \dots, u_n)$ is free, as it generates F , it is a basis of F , so its cardinality n is the dimension of F : $n = rk(F)$.

Conversely, if $n = rk(F)$, then $n = \dim F$, so the family $(u_1, u_2, \dots, u_n)$ generates F . Then it forms a basis, which shows that the family $(u_1, u_2, \dots, u_n)$ is free.

2. If the family generates E , then $E = F$, hence $\dim E = \dim F$ and $rk(F) = \dim E$

If $rk(F) = \dim E$, then $\dim E = \dim F$ and since we also know that $F \subset E$, we obtain $E = F$, therefore $E = Vect(u_1, u_2, \dots, u_n)$, then the family $(u_1, u_2, \dots, u_n)$ is a generator of E .

1.8 Exercises

Exercise 1.1

Determine which of the sets F_1, F_2 and F_3 are vector subspaces of $\mathbb{R}^3$.

$$F_1 = \{(x, y, z) \in \mathbb{R}^3 / x + y - z = 0\}$$

$$F_2 = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 0\}$$

$$F_3 = \{(x, y, z) \in \mathbb{R}^3 / xy + 2z = 0\}$$

$$F_4 = \{(x, y, z) \in \mathbb{R}^3 / x - y + z + 1 = 0\}$$

Solution

1. $F_1 = \{(x, y, z) \in \mathbb{R}^3 / x + y - z = 0\}$

(i). F_1 contains the zero vector, as $0_{\mathbb{R}^3} = (0, 0, 0) \in F_1$.

(ii). Let $u = (x, y, z), v = (x', y', z') \in F_1$, $u + v = (x + x', y + y', z + z') \in F_1$?

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= (0) + (0) = 0, \end{aligned}$$

so, $u + v \in F_1$.

(iii). Let $u = (x, y, z) \in F_1$, $\lambda \in \mathbb{K} = \mathbb{R}$, $\lambda u = (\lambda x, \lambda y, \lambda z) \in F_1$?

$$\begin{aligned} (\lambda x) + (\lambda y) - (\lambda z) &= \lambda(x + y - z) \\ &= \lambda(0) = 0, \end{aligned}$$

so, $\lambda u \in F_1$

Thus, F_1 is a subspace.

2. $F_2 = \{(x, y, z) \in \mathbb{R}^3 / x^2 + y^2 + z^2 = 0\}$

We have,

$$F_2 = \{(0, 0, 0)\} = \{0_{\mathbb{R}^3}\},$$

Thus, F_2 is a subspace.

3. $F_3 = \{(x, y, z) \in \mathbb{R}^3 / xy + 2z = 0\}$

For, $\lambda = -1$ and $u = (2, 1, -1)$, we have

$$u \in F_3, \text{ because } 2 \times 1 + 2 \times (-1) = 0,$$

but

$$\lambda u = (-2, -1, 1) \notin F_3.$$

Thus, F_3 is not a subspace.

$$4. F_4 = \{(x, y, z) \in \mathbb{R}^3 / x - y + z + 1 = 0\}$$

F_4 is not a subspace of $\mathbb{R}^3$, since $0_{\mathbb{R}^3} = (0, 0, 0) \notin F_4$.

Exercise 1.2

We consider the following families of vectors of $\mathbb{R}^3$:

$$B_1 = \{u_1 = (1, 1, 3), u_2 = (1, 2, 0)\},$$

$$B_2 = \{v_1 = (1, 0, -1), v_2 = (2, 1, 1), v_3 = (3, 0, 0)\}$$

$$B_3 = \{w_1 = (1, 1, 2), w_2 = (1, 3, 1), w_3 = (0, 0, 1), w_4 = (2, 4, 0)\}$$

Determine the free families, the generating families of $\mathbb{R}^3$ and the bases of $\mathbb{R}^3$.

Solution

$$1. B_1 = \{u_1 = (1, 1, 3), u_2 = (1, 2, 0)\}$$

- Is family B_1 free?

Consider $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $\alpha_1 u_1 + \alpha_2 u_2 = 0_{\mathbb{R}^3}$.

Then,

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 = 0 \\ 3\alpha_1 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0, \end{aligned}$$

thus, u_1 and u_2 are linearly independent. Then the family B_1 is free.

- Is family B_1 generative?

We have

$$\text{Card}(B_1) = 2 < \dim(\mathbb{R}^3),$$

then the family B_1 is not generative.

Is family B_1 base of $\mathbb{R}^3$?

Since, the family B_1 is not generative, then it is not a base of $\mathbb{R}^3$.

$$2. B_2 = \{v_1 = (1, 0, -1), v_2 = (2, 1, 1), v_3 = (3, 0, 0)\}$$

- Is family B_2 free?

Consider $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3}$.

Then,

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0 \\ \alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0, \end{aligned}$$

thus, u_1, u_2 and u_3 are linearly independent. Then the family B_2 is free.

- Is family B_2 generative?

Let, $X = (x, y, z) \in \mathbb{R}^3$, we solve the following vector equation $X = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$, with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

We have,

$$\begin{aligned} X = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 &\Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 + 3\alpha_3 = x \\ \alpha_2 = y \\ -\alpha_1 + \alpha_2 = z \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha_3 = \frac{x - \alpha_1 - 2\alpha_2}{3} = \frac{x - y + z - 2y}{3} = \frac{1}{3}x - y + \frac{1}{3}z \\ \alpha_2 = y \\ \alpha_1 = \alpha_2 - z = y - z \end{cases} \end{aligned}$$

Then, for all $X = (x, y, z) \in \mathbb{R}^3$, there exists $\alpha_1 = y - z, \alpha_2 = y, \alpha_3 = \frac{1}{3}x - y + \frac{1}{3}z \in \mathbb{R}$, such that $X = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$.

Then the family B_2 is generative.

Is family B_2 base of $\mathbb{R}^3$?

Since, the family B_2 is generative and free, then it is a base of $\mathbb{R}^3$.

3. $B_3 = \{w_1 = (1, 1, 2), w_2 = (1, 3, 1), w_3 = (0, 0, 1), w_4 = (2, 4, 0)\}$

- Is family B_3 free?

Consider $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, such that $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0_{\mathbb{R}^3}$.

Then,

$$\begin{aligned}
 \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
 &\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + 2\alpha_4 = 0 \\ \alpha_1 + 3\alpha_2 + 4\alpha_4 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -\alpha_2 - 2\alpha_4 \\ -\alpha_2 - 2\alpha_4 + 3\alpha_2 + 4\alpha_4 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \\
 &\Rightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_2 = -\alpha_4 \\ \alpha_3 = -3\alpha_4 \end{cases},
 \end{aligned}$$

we take $\alpha_4 = -1$, we find $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = 3$. Then there exists $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 3, -1) \neq (0, 0, 0, 0)$ such that, $\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 = 0_{\mathbb{R}^3}$.

Thus, w_1, w_2, w_3 and w_4 are linearly dependent. Then the family B_3 is not free.

- Is family B_3 generative?

Since

$$w_1 + w_2 + 3w_3 - w_4 = 0_{\mathbb{R}^3},$$

then,

$$\text{Span}(w_1, w_2, w_3, w_4) = \text{Span}(w_1, w_2, w_3)$$

Let, $X = (x, y, z) \in \mathbb{R}^3$, we solve the following vector equation $X = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$, with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

We have,

$$\begin{aligned}
 X = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 &\Leftrightarrow \begin{cases} \alpha_1 + \alpha_2 = x \\ \alpha_1 + 3\alpha_2 = y \\ 2\alpha_1 + \alpha_2 + \alpha_3 = z \end{cases} \\
 &\Leftrightarrow \begin{cases} \alpha_1 = \frac{3x-y}{2} \\ \alpha_2 = \frac{-x+y}{2} \\ \alpha_3 = \frac{-5x+y+2z}{2} \end{cases}
 \end{aligned}$$

Then, for all $X = (x, y, z) \in \mathbb{R}^3$, there exists $\alpha_1 = \frac{3x-y}{2}, \alpha_2 = \frac{-x+y}{2}, \alpha_3 = \frac{-5x+y+2z}{2} \in \mathbb{R}$, such that $X = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3$.

Then the family B_3 is generative.

- Is family B_3 base of $\mathbb{R}^3$?

Since, the family B_3 is not free, then it is not a base of $\mathbb{R}^3$.

Exercise 1.3

In $\mathbb{R}^3$, let $E = \{(x, y, z) \in \mathbb{R}^3 / x + y - 2z = 0 \text{ and } y - z = 0\}$ and F the subspace generated by $u = (1, 2, 3)$, $v = (1, 0, -2)$ and $w = (0, 4, 1)$.

1. Prove that E is a subspace of $\mathbb{R}^3$.
2. Determine a base and the dimension of E .
3. Determine a base and the dimension of $F \cap E$.

Solution**1. Prove that E is a subspace of $\mathbb{R}^3$.**

(i). E contains the zero vector, as $0_{\mathbb{R}^3} = (0, 0, 0) \in E$.

(ii). Let $u = (x, y, z), v = (x', y', z') \in E$, $u + v = (x + x', y + y', z + z') \in E$?

We have

$$\begin{aligned} (x + x') + (y + y') - 2(z + z') &= (x + y - 2z) + (x' + y' - 2z') \\ &= (0) + (0) = 0, \end{aligned}$$

and

$$\begin{aligned} (y + y') - (z + z') &= (y - z) + (y' - z') \\ &= (0) + (0) = 0, \end{aligned}$$

so, $u + v \in E$.

(iii). Let $u = (x, y, z) \in E$, $\lambda \in \mathbb{K} = \mathbb{R}$, $\lambda u = (\lambda x, \lambda y, \lambda z) \in E$?

We have

$$\begin{aligned} (\lambda x) + (\lambda y) - 2(\lambda z) &= \lambda(x + y - 2z) \\ &= \lambda(0) = 0, \end{aligned}$$

and

$$\begin{aligned} (\lambda y) - (\lambda z) &= \lambda(y - z) \\ &= \lambda(0) = 0, \end{aligned}$$

so, $\lambda u \in E$

Thus, E is a subspace.

2. Determine a base and the dimension of E .

Let $X = (x, y, z) \in E$, we have

$$\begin{cases} x + y - 2z = 0 \\ y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y + 2z = z \\ y = z \end{cases},$$

so,

$$X = (x, y, z) = (z, z, z) = z \underbrace{(1, 1, 1)}_{X_1},$$

then, E is the subspace generated with the vector $X_1 = (1, 1, 1)$. Since $X_1 \neq 0_{\mathbb{R}^3}$, then the family $B = \{X_1 = (1, 1, 1)\}$ forms a base of E .

3. Determine a base and the dimension of $F \cap E$.

Let $X = (x, y, z) \in \mathbb{R}^3$,

$$X \in F \cap E \Leftrightarrow X \in F \text{ and } X \in E,$$

we solve the equation $X_E = X_F$.

We have :

$$X \in E \Leftrightarrow X = (x, y, z) = \lambda (1, 1, 1)$$

and

$$X \in F \Leftrightarrow X = (x, y, z) = \alpha u + \beta v + \gamma w$$

$$\Leftrightarrow X = (\alpha + \beta, 2\alpha + 4\gamma, 3\alpha - 2\beta + \gamma)$$

So, we obtain the following system :

$$\begin{aligned} \begin{cases} \alpha + \beta = \lambda \\ 2\alpha + 4\gamma = \lambda \\ 3\alpha - 2\beta + \gamma = \lambda \end{cases} &\Leftrightarrow \begin{cases} \alpha = \lambda - \beta \\ 2(\lambda - \beta) + 4\gamma = \lambda \\ 3(\lambda - \beta) - 2\beta + \gamma = \lambda \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = \lambda - \beta \\ -2\beta + 4\gamma = -\lambda \\ -5\beta + \gamma = -2\lambda \end{cases} \Leftrightarrow \begin{cases} \alpha = \lambda - \beta \\ -2\beta + 4(-2\lambda + 5\beta) = -\lambda \\ \gamma = -2\lambda + 5\beta \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = \frac{11}{18}\lambda \\ \beta = \frac{7}{18}\lambda \\ \gamma = -\frac{1}{18}\lambda \end{cases} \end{aligned}$$

then,

$$X = \lambda (1, 1, 1),$$

so,

$$E \cap F = E = \text{Span}(X_1 = (1, 1, 1)).$$

Exercise 1.4

Let $E = \{(x, y, z) \in \mathbb{R}^3 / x + y - 2z = 0 \text{ and } 2x - y - z = 0\}$,

$F = \{(x, y, z) \in \mathbb{R}^3 / x + y - z = 0\}$ two subsets of $\mathbb{R}^3$ and $u = (1, 1, 1)$, $v = (1, 0, 1)$, $w = (0, 1, 1)$.

1. Prove that E and F are two subspaces of $\mathbb{R}^3$.
2. Determine a generating family of E and prove that this family is a base.
3. Prove that $\{v, w\}$ is a base of F .
4. Do we have $E \oplus F = \mathbb{R}^3$.

Solution

1. Prove that E is a subspaces of $\mathbb{R}^3$.

(i). E contains the zero vector, as $0_{\mathbb{R}^3} = (0, 0, 0) \in E$.

(ii). Let $u = (x, y, z)$, $v = (x', y', z') \in E$, $\alpha u + \beta v = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in E$?

We have

$$\begin{aligned} (\alpha x + \beta x') + (\alpha y + \beta y') - 2(\alpha z + \beta z') &= \alpha(x + y - 2z) + \beta(x' + y' - 2z') \\ &= (0) + (0) = 0, \end{aligned}$$

and

$$\begin{aligned} 2(\alpha x + \beta x') - (\alpha y + \beta y') - (\alpha z + \beta z') &= \alpha(2x - y - z) + \beta(2x' - y' - z') \\ &= (0) + (0) = 0, \end{aligned}$$

so, $\alpha u + \beta v \in E$.

Thus, E is a subspace.

Prove that F is a subspaces of $\mathbb{R}^3$.

(i). F contains the zero vector, as $0_{\mathbb{R}^3} = (0, 0, 0) \in F$.

(ii). Let $u = (x, y, z)$, $v = (x', y', z') \in F$, $\alpha u + \beta v = (\alpha x + \beta x', \alpha y + \beta y', \alpha z + \beta z') \in F$?

We have

$$\begin{aligned} (\alpha x + \beta x') + (\alpha y + \beta y') - (\alpha z + \beta z') &= \alpha(x + y - z) + \beta(x' + y' - z') \\ &= (0) + (0) = 0, \end{aligned}$$

so, $\alpha u + \beta v \in F$.

Thus, F is a subspace.

2. Determine a generating family of E and prove that this family is a base.

Let $X = (x, y, z) \in E$, we have

$$\begin{cases} x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ z = y \end{cases},$$

so,

$$X = (x, y, z) = (z, z, z) = z \underbrace{(1, 1, 1)}_u,$$

then, E is the subspace generated with the vector $u = (1, 1, 1)$. Since $u \neq 0_{\mathbb{R}^3}$, then the family $B = \{u = (1, 1, 1)\}$ forms a base of E .

3. Prove that $\{v, w\}$ is a base of F .

Let $X = (x, y, z) \in F$, we have

$$x + y - z = 0 \Leftrightarrow z = x + y,$$

so,

$$X = (x, y, x + y) = (x, 0, x) + (y, y, 0) = x \underbrace{(1, 0, 1)}_v + x \underbrace{(0, 1, 1)}_w,$$

then, F is the subspace generated with the vectors $v = (1, 0, 1)$ and $w = (0, 1, 1)$. Since $\{v, w\}$ are linearly independant, then the family $B' = \{v = (1, 0, 1), w = (0, 1, 1)\}$ forms a base of F .

4. Do we have $E \oplus F = \mathbb{R}^3$.

We have, $\dim E + \dim F = 3 = \dim \mathbb{R}^3$ and the family $B \cup B' = \{u = (1, 1, 1), v = (1, 0, 1), w = (0, 1, 1)\}$ forms a base of $\mathbb{R}^3$, then $E \oplus F = \mathbb{R}^3$.

Exercise 1.5

We consider the subspace of $\mathbb{R}^3$ defined by :

$$E = \{(x, y, z) \in \mathbb{R}^3 / x + y - 2z = 0 \text{ et } x - 3z = 0\}$$

1. Determine a base of E and specify the dimension of E .

2. Determine $\dim F$ such that F is the subspace of $\mathbb{R}^3$ generated by vectors

$$u_1 = (1, 2, 2), u_2 = (0, 6, -2), u_3 = (1, 5, 1)$$

3. Determine $E \cap F$.

4. Do we have $E \oplus F = \mathbb{R}^3$?

Solution

1. Determine a base of E

Let $X = (x, y, z) \in E$, we have

$$\begin{cases} x + y - 2z = 0 \\ y - 3z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -z \\ y = 3z \end{cases},$$

so,

$$X = (x, y, z) = (-z, 3z, z) = z \underbrace{(-1, 3, 1)}_{X_1},$$

then, E is the subspace generated with the vector $X_1 = (-1, 3, 1)$. Since $X_1 \neq 0_{\mathbb{R}^3}$, then the family $B = \{X_1 = (-1, 3, 1)\}$ forms a base of E .

The dimension of E .

$$\dim(E) = \text{Card}(B) = 1.$$

2. Determine $\dim F$ such that F is the subspace of $\mathbb{R}^3$ generated by vectors

$$u_1 = (1, 2, 2), u_2 = (0, 6, -2), u_3 = (1, 5, 1)$$

The family $B' = \{u_1 = (1, 2, 2), u_2 = (0, 6, -2), u_3 = (1, 5, 1)\}$ is generating F .

- Is family B' free?

Consider $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3}$.

Then,

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 6 \\ -2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} \alpha_1 + \alpha_3 = 0 \\ 2\alpha_1 + 6\alpha_2 + 5\alpha_3 = 0 \\ 2\alpha_1 - 2\alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -\alpha_3 \\ -2\alpha_3 + 6\alpha_2 + 5\alpha_3 = 0 \\ -2\alpha_3 - 2\alpha_2 + \alpha_3 = 0 \end{cases} \\ &\Rightarrow \begin{cases} \alpha_1 = -\alpha_3 \\ 6\alpha_2 + 3\alpha_3 = 0 \\ -2\alpha_2 - \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -\alpha_3 \\ \alpha_2 = -\frac{1}{2}\alpha_3 \end{cases} \end{aligned}$$

we take $\alpha_3 = -2$, we find $\alpha_1 = 2$ and $\alpha_2 = 1$. Then there exists $(\alpha_1, \alpha_2, \alpha_3) = (2, 1, -2) \neq (0, 0, 0)$ such that, $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3}$.

Thus, u_1, u_2 and u_3 are linearly dependent. Then the family B' is not free.

We choose the family $B'' = \{u_1 = (1, 2, 2), u_2 = (0, 6, -2)\}$, it is free, because for any $\alpha_1, \alpha_2 \in \mathbb{R}$, we have $\alpha_1 u_1 + \alpha_2 u_2 = 0_{\mathbb{R}^3}$. Thus the family B'' is a base of F .

Then,

$$\dim(F) = \text{Card}(B'') = 2.$$

3. Determine $E \cap F$.

Let $X = (x, y, z) \in \mathbb{R}^3$,

$$X \in F \cap E \Leftrightarrow X \in F \text{ and } X \in E,$$

we solve the equation $X_E = X_F$.

We have :

$$X \in E \Leftrightarrow X = (x, y, z) = \lambda(-1, 3, 1)$$

and

$$\begin{aligned} X \in F &\Leftrightarrow X = (x, y, z) = \alpha u + \beta v \\ &\Leftrightarrow X = (\alpha, 2\alpha + 6\beta, 2\alpha - 2\beta) \end{aligned}$$

So, we obtain the following system :

$$\begin{aligned} \begin{cases} \alpha = -\lambda \\ 2\alpha + 6\beta = 3\lambda \\ 2\alpha - 2\beta = \lambda \end{cases} &\Leftrightarrow \begin{cases} \alpha = -\lambda \\ -2\lambda + 6\beta = 3\lambda \\ -2\lambda - 2\beta = \lambda \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = -\lambda \\ \beta = \frac{5}{6}\lambda \\ 3\lambda + 2\beta = 0 \end{cases} \Leftrightarrow \alpha = \beta = \lambda = 0, \end{aligned}$$

then,

$$E \cap F = \{(0, 0, 0)\} = \{0_{\mathbb{R}^3}\}.$$

4. Do we have $E \oplus F = \mathbb{R}^3$?

We have :

(i). $E \cap F = \{0_{\mathbb{R}^3}\}.$

(ii). $E + F = \mathbb{R}^3?$

Let $X = (x, y, z) \in \mathbb{R}^3$, we solve the vector equation $\alpha X_1 + \beta u + \gamma v = X$ with $\alpha, \beta, \gamma \in \mathbb{R}$.

Or we prove that the family $B \cup B'' = \{X_1, u, v\}$ is a base of $\mathbb{R}^3$.

$$\begin{aligned}
 \alpha X_1 + \beta u + \gamma v = X &\Leftrightarrow \alpha \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
 &\Leftrightarrow \begin{cases} -\alpha + \beta = x \\ 3\alpha + 2\beta + 6\gamma = y \\ \alpha + 2\beta - 2\gamma = z \end{cases} \Leftrightarrow \begin{cases} \beta = \alpha + x \\ 3\alpha + 2(\alpha + x) + 6\gamma = y \\ \alpha + 2(\alpha + x) - 2\gamma = z \end{cases} \\
 &\Leftrightarrow \begin{cases} \beta = \alpha + x \\ 5\alpha + 6\gamma = -2x + y \\ 3\alpha - 2\gamma = -2x + z \end{cases} \Leftrightarrow \begin{cases} \beta = \alpha + x \\ \alpha = \frac{-2x+y-6\gamma}{5} \\ 3\left(\frac{-2x+y-6\gamma}{5}\right) - 2\gamma = -2x + z \end{cases} \\
 &\Leftrightarrow \begin{cases} \beta = \frac{6}{14}x + \frac{1}{14}y + \frac{3}{14}z \\ \alpha = -\frac{8}{14}x + \frac{1}{14}y + \frac{3}{14}z \\ \gamma = \frac{4}{28}x + \frac{3}{28}y - \frac{5}{28}z \end{cases},
 \end{aligned}$$

then, for all $X = (x, y, z) \in \mathbb{R}^3$, there exists $\alpha = -\frac{8}{14}x + \frac{1}{14}y + \frac{3}{14}z, \beta = \frac{6}{14}x + \frac{1}{14}y + \frac{3}{14}z, \gamma = \frac{4}{28}x + \frac{3}{28}y - \frac{5}{28}z \in \mathbb{R}$, such that $\alpha X_1 + \beta u + \gamma v = X$.

So, $E + F = \mathbb{R}^3$. Thus $E \oplus F = \mathbb{R}^3$.

1.9 Terminology translation

English	Frensh
Vector	Vecteur
Space	Espace
Subspace	Sous-espace
Field	Corps
Rank	Rang
Internal composition law	Loi de composition interne
External composition law	Loi de composition externe
Generating family	Famille génératrice
Basis	Base
Neutral element	Élément neutre
Scalar	Scalaire
Non-empty	Non vide
Subset	Sous-ensemble
Linear combination	Combinaison linéaire
Direct Sum	Somme directe
Supplementary	Supplémentaire
Linearly independent	Linéqirement indépendant
Free family	Famille libre

Linear applications

2.1 Definitions and properties

Definition 2.1

Let E, F be two $\mathbb{k}$ -vector spaces.

A linear application from E to F is an application $f : E \rightarrow F$, with the following properties :

1. $\forall u, v \in E$,

$$f(u + v) = f(u) + f(v).$$

2. $\forall u \in E, \forall \lambda \in \mathbb{k}$,

$$f(\lambda u) = \lambda f(u).$$

- A bijective linear application is called an isomorphism.
- A linear application from E to itself is called an endomorphism of E .
- A bijective endomorphism is called an automorphism.
- The set of linear applications from E to F is denoted by $L_{\mathbb{k}}(E, F)$ or $L(E, F)$.
- If $E = F$, the linear application is called endomorphism. The set of endomorphism of E is denoted by $L_{\mathbb{k}}(E)$ or $L(E)$.

Theorem 2.1

Let E, F be two $\mathbb{k}$ -vector spaces and let $f : E \rightarrow F$ be an application. For f to be a linear application, it is necessary and sufficient that :

$$\forall u, v \in E, \forall \alpha, \beta \in \mathbb{k}, f(\alpha u + \beta v) = \alpha f(u) + \beta f(v).$$

Proof

1. Suppose f is a linear application. Then, for all $u, v \in E$, and for all $\alpha, \beta \in \mathbb{k}$, we have

$$\begin{aligned} f\left(\underbrace{\alpha u}_X + \underbrace{\beta v}_Y\right) &= f(X + Y) = f(X) + f(Y) \\ &= f(\alpha u) + f(\beta v) = \alpha f(u) + \beta f(v). \end{aligned}$$

2. Conversely, if f satisfies the given condition for all $u, v \in E$, and for all $\alpha, \beta \in \mathbb{k}$, then

- For $\alpha = \beta = 1_{\mathbb{k}}$, we have

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \Leftrightarrow f(u + v) = f(u) + f(v).$$

- For $\alpha = \lambda$ and $\beta = 0$, we have

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \Leftrightarrow f(\lambda u) = \lambda f(u).$$

Example 2.1

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be an application defined by : $f(x, y) = (x + y, 2x, x - y)$.

For $u = (x, y), v = (x', y') \in \mathbb{R}^2$ and $\lambda \in \mathbb{k} = \mathbb{R}$, we have

(i).

$$\begin{aligned} f(u + v) &= f(x + x', y + y') \\ &= (x + x' + y + y', 2(x + x'), x + x' - (y + y')) \\ &= (x + y, 2x, x - y) + (x' + y', 2x', x' - y') \\ &= f(x, y) + f(x', y') = f(u) + f(v) \end{aligned}$$

(ii).

$$\begin{aligned} f(\lambda u) &= f(\lambda x, \lambda y) \\ &= (\lambda x + \lambda y, 2\lambda x, \lambda x - \lambda y) \\ &= \lambda(x + y, 2x, x - y) = \lambda f(x, y) = \lambda f(u), \end{aligned}$$

from (i) and (ii) we deduce that f is a linear application.

Examples 2.2

Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an application defined by : $g(x, y) = (x + y, 2xy)$.

For $u = (1, 2), v = (2, 2) \in \mathbb{R}^2$, we have

$$g(u + v) = g(3, 4) = (7, 24),$$

and

$$g(u) + g(v) = (3, 4) + (4, 8) = (7, 12),$$

then, $\exists u = (1, 2), v = (2, 2) \in \mathbb{R}^2$, such that $g(u + v) \neq g(u) + g(v)$. So, g is not a linear application.

Theorem 2.2

Let f be a linear application. Then we have :

$$f(0_E) = 0_F$$

and

$$f(-u) = -f(u), \text{ for all } u \in E.$$

Remark 2.1

If $f(0_E) \neq 0_F$, then f is not a linear application.

Examples 2.3

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an application defined by : $f(x, y) = (x + y, x - y + 1)$.

We have

$$f(0_{\mathbb{R}^2}) = f(0, 0) = (0, 1) \neq 0_{\mathbb{R}^2},$$

then, f is not a linear application.

Property 2.1

Let f, g be two linear applications from E to F and $\lambda \in \mathbb{K}$. The applications $f + g$ and λf defined by $(f + g)(u) = f(u) + g(u)$ and $(\lambda f)(u) = \lambda f(u)$ are linear.

Proof

Let u, v a vectors of E and α, β a scalars of $\mathbb{K}$. So, we have :

$$\begin{aligned} (f + g)(\alpha u + \beta v) &= f(\alpha u + \beta v) + g(\alpha u + \beta v) \\ &= \alpha f(u) + \beta f(v) + \alpha g(u) + \beta g(v) \\ &= \alpha(f(u) + g(u)) + \beta(f(v) + g(v)) \\ &= \alpha(f + g)(u) + \beta(f + g)(v), \end{aligned}$$

then $f + g$ is a linear application.

And we also have :

$$\begin{aligned}
 (\lambda f)(\alpha u + \beta v) &= \lambda f(\alpha u + \beta v) \\
 &= \lambda(\alpha f(u) + \beta f(v)) \\
 &= \alpha(\lambda f(u)) + \beta(\lambda f(v)) \\
 &= \alpha(\lambda f)(u) + \beta(\lambda f)(v),
 \end{aligned}$$

then λf is a linear application.

Property 2.2

Let $f : E \rightarrow F, g : F \rightarrow G$ be two linear applications. The applications $g \circ f$ defined by $(g \circ f)(u) = g(f(u))$ is linear.

Proof

Let u, v a vectors of E and α, β a scalars of $\mathbb{k}$. So, we have :

$$\begin{aligned}
 (g \circ f)(\alpha u + \beta v) &= g(f(\alpha u + \beta v)) = g(\alpha f(u) + \beta f(v)) \\
 &= \alpha g(f(u)) + \beta g(f(v)) \\
 &= \alpha(g \circ f)(u) + \beta(g \circ f)(v),
 \end{aligned}$$

then $g \circ f$ is a linear application.

2.2 Image and kernel of a linear application

Proposition 2.1

Let E, F be two $\mathbb{k}$ -vector space and $f \in L(E, F)$.

1. For any vector subspace F_1 of F , the inverse image $f^{-1}(F_1)$ is a vector subspace of E .
2. For any vector subspace E_1 of E , the direct image $f(E_1)$ is a vector subspace of F .

Proof

1. Let F_1 a vector subspace of F

(1.a). $f^{-1}(F_1) \neq \emptyset$: $f(0_E) = 0_F$, then $0_E \in f^{-1}(F_1)$.

(1.b). Let $\alpha, \beta \in \mathbb{k}$ and $u, v \in f^{-1}(F_1)$. We have

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \in F_1 \Rightarrow \alpha u + \beta v \in f^{-1}(F_1),$$

from (1.a) and (1.b), we deduce that $f^{-1}(F_1)$ is a vector subspace of E .

2. Let E_1 a vector subspace of E

(2.a). $f(E_1) \neq \emptyset$: $f(0_E) = 0_F$, then $0_F \in f(E_1)$.

(2.b). Let $\alpha, \beta \in \mathbb{K}$ and $u, v \in f(E_1)$. Then, there exists $X, Y \in E$, such that $u = f(X)$ and $v = f(Y)$. We have

$$\alpha u + \beta v = \alpha f(X) + \beta f(Y) = f(\alpha X + \beta Y) \in f(E_1),$$

from (2.a) and (2.b), we deduce that $f(E_1)$ is a vector subspace of F .

Definition 2.2

Let f be a linear application from E to F .

- The set :

$$\ker(f) = f^{-1}(\{0_F\}) = \{u \in E / f(u) = 0_F\},$$

is called the kernel of the linear application f .

- The set :

$$\text{Im}(f) = f(E) = \{f(u) / u \in E\},$$

is called the image of the linear application f .

Examples 2.4

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear application defined by : $f(x, y, z) = (x + y - z, x + z)$.

- **The kernel of f**

$$\ker(f) = f^{-1}(\{0_{\mathbb{R}^2}\}) = \{u = (x, y, z) \in \mathbb{R}^3 / f(u) = 0_{\mathbb{R}^2}\}.$$

We solve the vector equation $f(u) = 0_{\mathbb{R}^2}$.

$$f(u) = 0_{\mathbb{R}^2} \Leftrightarrow \begin{cases} x + y - z = 0 \\ x + z = 0 \end{cases} \Leftrightarrow \begin{cases} y = -x + z = 2z \\ x = -z \end{cases}$$

then,

$$u = (x, y, z) = (-z, 2z, z) = z \underbrace{(-1, 2, 1)}_{X_1},$$

so, $\ker f$ is the vector subspace generated by the vector $X_1 = (-1, 2, 1)$.

$$\ker f = \text{Span}(X_1 = (-1, 2, 1)).$$

- The image of f

$$\begin{aligned}
\text{Im}(f) &= \{f(u) / u = (x, y, z) \in \mathbb{R}^3\} \\
&= \{(x + y - z, x + z) / (x, y, z) \in \mathbb{R}^3\} \\
&= \{(x, x) + (y, 0) + (-z, z) / (x, y, z) \in \mathbb{R}^3\} \\
&= \left\{ \underbrace{x(1, 1)}_{Y_1} + \underbrace{y(1, 0)}_{Y_2} + \underbrace{z(-1, 1)}_{Y_3} / (x, y, z) \in \mathbb{R}^3 \right\}.
\end{aligned}$$

Hence, $\text{Im } f$ is the vector subspace of $\mathbb{R}^2$ spanned by the family $B = \{Y_1 = (1, 1), Y_2 = (1, 0), Y_3 = (-1, 1)\}$, thus :

$$\text{Im } f = \text{span}(Y_1 = (1, 1), Y_2 = (1, 0), Y_3 = (-1, 1)).$$

Note that the vectors Y_1 and Y_2 are linearly independent, while Y_3 depends on them. Therefore :

$$\text{Im } f = \text{span}(Y_1 = (1, 1), Y_2 = (1, 0)).$$

Proposition 2.2

Let E, F be two $\mathbb{k}$ -vector spaces and $f \in L(E, F)$.

1. f is injective if and only if $\ker(f) = \{0_E\}$
2. f is surjective if and only if $\text{Im}(f) = F$.

Proof

1. ($\Rightarrow$) Suppose that f is injective. Let $u \in \ker f$, then

$$f(u) = 0_F = f(0_E),$$

since f is injective, then $u = 0_E$. Thus $\ker(f) = \{0_E\}$.

- ($\Leftarrow$) Suppose that $\ker(f) = \{0_E\}$. Let $u, v \in E$

$$f(u) = f(v) \Rightarrow f(u) - f(v) = 0_F$$

$$\Rightarrow f(u - v) = 0_F$$

$$\Rightarrow u - v \in \ker f$$

$$\Rightarrow u - v = 0_E \Rightarrow u = v;$$

then f is injective.

2.

$$\begin{aligned} f \text{ is surjective} &\Leftrightarrow \forall Y \in F, \exists X \in E, Y = f(X) \\ &\Leftrightarrow f(E) = F \\ &\Leftrightarrow \text{Im } f = F. \end{aligned}$$

2.3 Rank of a linear application

Definition 2.3

Let E and F be two finite-dimensional $\mathbb{k}$ -vector spaces, and f be a linear application from E to F .

We call the rank of f , denoted $\text{rank}(f)$, the natural number defined by :

$$\text{rank}(f) = \dim(\text{Im } f).$$

Examples 2.5

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear application defined by : $f(x, y, z) = (y + z, x + z, x + y)$.

- The image of f

$$\begin{aligned} \text{Im}(f) &= \{f(u) / u = (x, y, z) \in \mathbb{R}^3\} \\ &= \{(y + z, x + z, x + y) / (x, y, z) \in \mathbb{R}^3\} \\ &= \{(0, x, x) + (y, 0, y) + (z, z, 0) / (x, y, z) \in \mathbb{R}^3\} \\ &= \left\{ x \underbrace{(0, 1, 1)}_{Y_1} + y \underbrace{(1, 0, 1)}_{Y_2} + z \underbrace{(1, 1, 0)}_{Y_3} / (x, y, z) \in \mathbb{R}^3 \right\}. \end{aligned}$$

Hence, $\text{Im } f$ is the vector subspace of $\mathbb{R}^3$ spanned by the family $B = \{Y_1 = (0, 1, 1), Y_2 = (1, 0, 1), Y_3 = (1, 1, 0)\}$, thus :

$$\text{Im } f = \text{span}(Y_1 = (0, 1, 1), Y_2 = (1, 0, 1), Y_3 = (1, 1, 0)).$$

Note that the vectors Y_1, Y_2 and Y_3 are linearly independent, because for all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{k} = \mathbb{R}$, we have :

$$\begin{aligned} \sum_{i=1}^3 \alpha_i Y_i = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{cases} \alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \\ &\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0, \end{aligned}$$

then the family B form a base to $\text{Im} f$. So,

$$\text{Rank}(f) = \dim(\text{Im } f) = \text{card}(B) = 3.$$

2.3.1 Rank theorem

Theorem 2.3

Let f be a linear application from E to F , then we have

1. f is surjective if and only if $\text{rank}(f) = \dim(F)$.
2. f is injective if and only if $\text{rank}(f) = \dim(E)$.

Theorem 2.4 Rank Theorem

Let f be a linear application from E to F . If E is finite-dimensional, then :

$$\text{rank}(f) + \dim(\ker(f)) = \dim(E).$$

Proof

Let $\dim E = n$ and $k = \dim \ker(f)$ and consider a basis $\{v_1, \dots, v_k\}$ of $\ker f$ as a subspace of E . We can always extend this basis, to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of the whole E .

We would like to prove that the family $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of the subspace $\text{Im}(f)$ of F .

1. Span : Let $Y \in \text{Im}(f)$. Then $Y = f(X)$, for some $X \in E$. As X is an element of E and $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis, we can find a unique set of coefficients $\alpha_i \in \mathbb{k}$ such that

$$X = \alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n$$

But then :

$$\begin{aligned}
 Y = f(X) &= f(\alpha_1 v_1 + \dots + \alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots + \alpha_n v_n) \\
 &= \alpha_1 f(v_1) + \dots + \alpha_k f(v_k) + \alpha_{k+1} f(v_{k+1}) + \dots + \alpha_n f(v_n) \\
 &= 0.f(v_1) + \dots + 0.f(v_k) + \alpha_{k+1} f(v_{k+1}) + \dots + \alpha_n f(v_n) \\
 &= \alpha_{k+1} f(v_{k+1}) + \dots + \alpha_n f(v_n)
 \end{aligned}$$

And we obtain that the system of vectors $\{f(v_{k+1}), \dots, f(v_n)\}$ span the image $\text{Im } f$.

2. Linearly independent :

Suppose that a linear combination of $\{f(v_{k+1}), \dots, f(v_n)\}$ gives 0. Say that we have

$$\beta_{k+1} f(v_{k+1}) + \dots + \beta_n f(v_n) = 0_F \Leftrightarrow f(\beta_{k+1} v_{k+1} + \dots + \beta_n v_n) = 0_F$$

and the element $X' = \beta_{k+1} v_{k+1} + \dots + \beta_n v_n$ will be in the kernel of f . Since $\{v_1, \dots, v_k\}$ is a basis of $\ker f$ and the whole system $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is linearly independent, the only possibility is that $X' = 0_E$ and $\beta_{k+1} = \beta_{k+2} = \dots = \beta_n = 0$.

We have therefore obtained that the vectors $\{f(v_{k+1}), \dots, f(v_n)\}$ are linearly independent in F .

As a combination of (1) and (2), we have proven that $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of $\text{Im } f$ and the dimension of $\text{Im } f$, which is the rank of f , is $n - k$. In any case, we have :

$$\text{rank}(f) + \dim(\ker(f)) = \dim(E).$$

2.4 Inverse of a linear application

Definition 2.4

Let E, F two $\mathbb{k}$ -vector spaces and $f \in L(E, F)$.

The application $f : E \rightarrow F$ is said to be invertible if, for all $Y \in F$, the equation $Y = f(X)$ admits a unique solution $X \in E$.

Definition 2.5

Let E, F two $\mathbb{k}$ -vector spaces and $f \in L(E, F)$ invertible. Then its reciprocal $f^{-1} : F \rightarrow E$ is defined by

$$f^{-1}(Y) = (\text{the unique } X \in E, \text{ such that } Y = f(X)),$$

and we have :

$$f^{-1} \circ f = Id_E \quad \text{and} \quad f \circ f^{-1} = Id_F$$

Examples 2.6

The linear application $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by : $f(x, y) = (x + y, x - y)$ is inversible, because, for all $Y = (a, b) \in F = \mathbb{R}^2$, the equation $Y = f(X)$ admets a unique solution.

$$Y = f(X) \Leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix} \Leftrightarrow \begin{cases} x + y = a \\ x - y = b \end{cases} \Leftrightarrow \begin{cases} x = \frac{a+b}{2} \\ y = \frac{a-b}{2} \end{cases},$$

then,

$$\begin{aligned} f^{-1} : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (a, b) &\mapsto \left(\frac{a+b}{2}, \frac{a-b}{2} \right) \end{aligned}$$

2.5 Exercises

Exercise 2.1

Determine whether the following applications are linear applications

1. $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \mapsto (x + y, y, x - 3y)$
2. $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x + y, x - 3z + 1)$
3. $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x^2 + y^2, y + z)$.

Solution

1. $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \mapsto (x + y, y, x - 3y)$

Let $u = (x, y), v = (x', y') \in \mathbb{R}^2$, and $\lambda \in \mathbb{k} = \mathbb{R}$, we have

(i).

$$\begin{aligned} f(u + v) &= f(x + x', y + y') = (x + x' + y + y', y + y', x + x' - 3(y + y')) \\ &= (x + y, y, x - 3y) + (x' + y', y', x' - 3y') \\ &= f(x, y) + f(x', y') = f(u) + f(v). \end{aligned}$$

(ii).

$$f(\lambda u) = f(\lambda x, \lambda y) = (\lambda x + \lambda y, \lambda y, \lambda x - 3\lambda y) = \lambda(x + y, y, x - 3y) = \lambda f(u),$$

from (i) and (ii), we deduce that f is a linear application.

$$2. f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x + y, x - 3z + 1)$$

We have

$$f(0_{\mathbb{R}^3}) = f(0, 0, 0) = (0, 1) \neq 0_{\mathbb{R}^2},$$

therefore, f is not a linear application.

$$3. f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x^2 + y^2, y + z).$$

For $u = (1, 2, 1)$ and $\lambda = 2$, we have

$$f(\lambda u) = f(2, 4, 2) = (20, 6)$$

and

$$\lambda f(u) = 2(5, 3) = (10, 6),$$

then, $\exists u = (1, 2, 1) \in \mathbb{R}^3, \exists \lambda = 2 \in \mathbb{k} = \mathbb{R}, f(\lambda u) \neq \lambda f(u)$, so f is not a linear application.

Exercise 2.2

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be an application defined for all $u = (x, y, z) \in \mathbb{R}^3$ by :

$$f(x, y, z) = (x + 2y - z, x + y + z)$$

1. Prove that f is linear.

2. Determine $\ker f$ et $\text{Im } f$.

Solution

1. Prove that f is linear.

Let $u = (x, y, z), v = (x', y', z') \in \mathbb{R}^3$, and $\lambda \in \mathbb{k} = \mathbb{R}$, we have

(i).

$$\begin{aligned} f(u + v) &= f(x + x', y + y', z + z') = (x + x' + 2y + 2y' - z - z', x + x' + y + y' + z + z') \\ &= (x + 2y - z, x + y + z) + (x' + 2y' - z', x' + y' + z') \\ &= f(x, y, z) + f(x', y', z') = f(u) + f(v). \end{aligned}$$

(ii).

$$f(\lambda u) = f(\lambda x, \lambda y, \lambda z) = (\lambda x + 2\lambda y - \lambda z, \lambda x + \lambda y + \lambda z) = \lambda(x + 2y - z, x + y + z) = \lambda f(u),$$

from (i) and (ii), we deduce that f is a linear application.

2. Determine $\ker f$ et $\operatorname{Im} f$.

- **The kernel of f**

$$\ker(f) = f^{-1}(\{0_{\mathbb{R}^2}\}) = \{u = (x, y, z) \in \mathbb{R}^3 / f(u) = 0_{\mathbb{R}^2}\}.$$

We solve the vector equation $f(u) = 0_{\mathbb{R}^2}$.

$$f(u) = 0_{\mathbb{R}^2} \Leftrightarrow \begin{cases} x + 2y - z = 0 \\ x + y + z = 0 \end{cases} \Leftrightarrow \begin{cases} z = \frac{1}{2}y \\ x = -\frac{3}{2}y \end{cases}$$

then,

$$u = (x, y, z) = \left(-\frac{3}{2}y, y, \frac{1}{2}y\right) = \frac{1}{2}y \underbrace{(-3, 2, 1)}_{X_1},$$

so, $\ker f$ is the vector subspace generated by the vector $X_1 = (-3, 2, 1)$.

$$\ker f = \operatorname{Span}(X_1 = (-3, 2, 1)).$$

- **The image of f**

$$\begin{aligned} \operatorname{Im}(f) &= \{f(u) / u = (x, y, z) \in \mathbb{R}^3\} \\ &= \{(x + 2y - z, x + y + z) / (x, y, z) \in \mathbb{R}^3\} \\ &= \{(x, x) + (2y, y) + (-z, z) / (x, y, z) \in \mathbb{R}^3\} \\ &= \left\{ \underbrace{x(1, 1)}_{Y_1} + \underbrace{y(2, 1)}_{Y_2} + \underbrace{z(-1, 1)}_{Y_3} / (x, y, z) \in \mathbb{R}^3 \right\}. \end{aligned}$$

Hence, $\operatorname{Im} f$ is the vector subspace of $\mathbb{R}^2$ spanned by the family $B = \{Y_1 = (1, 1), Y_2 = (2, 1), Y_3 = (-1, 1)\}$, thus :

$$\operatorname{Im} f = \operatorname{span}(Y_1 = (1, 1), Y_2 = (2, 1), Y_3 = (-1, 1)).$$

Note that the vectors Y_1 and Y_2 are linearly independent, while Y_3 depends on them. Therefore :

$$\operatorname{Im} f = \operatorname{span}(Y_1 = (1, 1), Y_2 = (2, 1)).$$

Exercise 2.3

Let f an endomorphism of $\mathbb{R}^3$ whose image of the canonical base $B_0 = \{e_1, e_2, e_3\}$ is

$$f(e_1) = e_1 + e_2 - e_3$$

$$f(e_2) = e_1 - e_2 + e_3$$

$$f(e_3) = 2e_1 + e_2 + e_3$$

1. For all $u = (x, y, z) \in \mathbb{R}^3$, determine $f(u)$.

2. Determine the kernel of f .

3. Give a base of the image of f .

Solution

1. For all $u = (x, y, z) \in \mathbb{R}^3$, determine $f(u)$.

Let $u = (x, y, z) \in \mathbb{R}^3$,

$$\begin{aligned} f(u) &= f(x, y, z) = f(xe_1 + ye_2 + ze_3) \\ &= xf(e_1) + yf(e_2) + zf(e_3) \\ &= x(e_1 + e_2 - e_3) + y(e_1 - e_2 + e_3) + z(2e_1 + e_2 + e_3) \\ &= (x + y + 2z)e_1 + (x - y + z)e_2 + (-x + y + z)e_3 \\ &= (x + y + 2z, x - y + z, -x + y + z) \end{aligned}$$

2. Determine the kernel of f .

$$\ker(f) = f^{-1}(\{0_{\mathbb{R}^3}\}) = \{u = (x, y, z) \in \mathbb{R}^3 / f(u) = 0_{\mathbb{R}^3}\}.$$

We solve the vector equation $f(u) = 0_{\mathbb{R}^3}$.

$$f(u) = 0_{\mathbb{R}^3} \Leftrightarrow \begin{cases} x + y + 2z = 0 \\ x - y + z = 0 \\ -x + y + z = 0 \end{cases} \Leftrightarrow x = y = z = 0,$$

then,

$$\ker f = \{(0, 0, 0)\} = \{0_{\mathbb{R}^3}\}.$$

3. Give a base of the image of f .

- The image of f

$$\begin{aligned}
\operatorname{Im}(f) &= \{f(u) / u = (x, y, z) \in \mathbb{R}^3\} \\
&= \{(x + y + 2z, x - y + z, -x + y + z) / (x, y, z) \in \mathbb{R}^3\} \\
&= \{(x, x, -x) + (y, -y, y) + (2z, z, z) / (x, y, z) \in \mathbb{R}^3\} \\
&= \left\{ x \underbrace{(1, 1, -1)}_{Y_1} + y \underbrace{(1, -1, 1)}_{Y_2} + z \underbrace{(2, 1, 1)}_{Y_3} / (x, y, z) \in \mathbb{R}^3 \right\}
\end{aligned}$$

Hence, $\operatorname{Im} f$ is the vector subspace of $\mathbb{R}^3$ spanned by the family $B = \{Y_1 = (1, 1, -1), Y_2 = (1, -1, 1), Y_3 = (2, 1, 1)\}$, thus :

$$\operatorname{Im} f = \operatorname{span}(Y_1 = (1, 1, -1), Y_2 = (1, -1, 1), Y_3 = (2, 1, 1)).$$

Note that the vectors Y_1, Y_2 and Y_3 are linearly independent, because for all $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{K} = \mathbb{R}$, we have :

$$\begin{aligned}
\sum_{i=1}^3 \alpha_i Y_i = 0_{\mathbb{R}^3} &\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \\
&\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0,
\end{aligned}$$

then the family B form a base to $\operatorname{Im} f$.

Exercise 2.4

We consider the linear application $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by :

$$f(x_1, x_2, x_3, x_4) = (2x_1 + x_2 - x_4, x_1 + 3x_2, x_3 - 4x_4)$$

Determine the kernel and image of f .

Solution**1. The kernel of f .**

$$\ker(f) = f^{-1}(\{0_{\mathbb{R}^3}\}) = \{u = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 / f(u) = 0_{\mathbb{R}^3}\}.$$

We solve the vector equation $f(u) = 0_{\mathbb{R}^3}$.

$$\begin{aligned} f(u) = 0_{\mathbb{R}^3} &\Leftrightarrow \begin{cases} 2x_1 + x_2 - x_4 = 0 \\ x_1 + 3x_2 = 0 \\ x_3 - 4x_4 = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} 2(-3x_2) + x_2 - x_4 = 0 \\ x_1 = -3x_2 \\ x_3 = 4x_4 \end{cases} \\ &\Leftrightarrow \begin{cases} x_4 = -5x_2 \\ x_1 = -3x_2 \\ x_3 = -20x_2 \end{cases} \end{aligned}$$

then,

$$u = (x_1, x_2, x_3, x_4) = (-3x_2, x_2, -20x_2, -5x_2) = x_2(-3, 1, -20, -5),$$

so, $\ker f$ is the vector subspace generated by the vector $u_1 = (-3, 1, -20, -5)$.

$$\ker f = \text{Span}(u_1 = (-3, 1, -20, -5)).$$

2. The image of f .

$$\begin{aligned} \text{Im}(f) &= \{f(u) / u = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\} \\ &= \{(2x_1 + x_2 - x_4, x_1 + 3x_2, x_3 - 4x_4) / (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\} \\ &= \{(2x_1, x_1, 0) + (x_2, 3x_2, 0) + (0, 0, x_3) + (-x_4, 0, -4x_4) / (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\} \\ &= \left\{ x_1 \underbrace{(2, 1, 0)}_{Y_1} + x_2 \underbrace{(1, 3, 0)}_{Y_2} + x_3 \underbrace{(0, 0, 1)}_{Y_3} - x_4 \underbrace{(1, 0, 4)}_{Y_4} / x_i \in \mathbb{R} \right\}. \end{aligned}$$

Hence, $\text{Im } f$ is the vector subspace of $\mathbb{R}^3$ spanned by the family $B = \{Y_1 = (2, 1, 0), Y_2 = (1, 3, 0), Y_3 = (0, 0, 1), Y_4 = (1, 0, 4)\}$, thus :

$$\text{Im } f = \text{span}(Y_1 = (2, 1, 0), Y_2 = (1, 3, 0), Y_3 = (0, 0, 1), Y_4 = (1, 0, 4)).$$

we see that, $\frac{3}{5}Y_1 - \frac{1}{5}Y_2 + 4Y_3 = Y_4$, then

$$\text{Im } f = \text{span}(Y_1 = (2, 1, 0), Y_2 = (1, 3, 0), Y_3 = (0, 0, 1)).$$

Exercise 2.5

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a linear application verifying :

$$f(1, 2, 4) = (3, -2, 1)$$

$$f(3, 0, 1) = (1, 4, 2)$$

$$f(0, 0, 2) = (0, -3, 5)$$

1. Calculate $f(5, 3, 7)$.
2. Determine $f(x, y, z)$ with $(x, y, z) \in \mathbb{R}^3$.

Solution**1. Calculate $f(5, 3, 7)$.**

We pose

$$B = \{u = (1, 2, 4), v = (3, 0, 1), w = (0, 0, 2)\},$$

it is easy to verify that family B forms a basis of $\mathbb{R}^3$.

Now we calculate the coordinates of the vector $X = (5, 3, 7)$ in the basis B .

$$X = \alpha u + \beta v + \gamma w \Leftrightarrow \begin{cases} \alpha + 3\beta = 5 \\ 2\alpha = 3 \\ 4\alpha + \beta + 2\gamma = 7 \end{cases} \Leftrightarrow \begin{cases} \beta = \frac{5-\alpha}{3} = \frac{7}{6} \\ \alpha = \frac{3}{2} \\ \gamma = \frac{7-4\alpha-\beta}{2} = -\frac{1}{12} \end{cases}$$

Then,

$$\begin{aligned} f(5, 3, 7) &= f\left(\frac{3}{2}u + \frac{7}{6}v - \frac{1}{12}w\right) = \frac{3}{2}f(u) + \frac{7}{6}f(v) - \frac{1}{12}f(w) \\ &= \frac{3}{2}(3, -2, 1) + \frac{7}{6}(1, 4, 2) - \frac{1}{12}(0, -3, 5) \\ &= \left(\frac{17}{3}, \frac{23}{12}, \frac{41}{12}\right). \end{aligned}$$

2. Determine $f(x, y, z)$ with $(x, y, z) \in \mathbb{R}^3$.

$$\begin{aligned} f(x, y, z) &= f((x, 0, 0) + (0, y, 0) + (0, 0, z)) \\ &= f(x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)) \\ &= xf(1, 0, 0) + yf(0, 1, 0) + zf(0, 0, 1) \end{aligned}$$

We calculate the coordinates of the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in the basis B .

$$(1, 0, 0) = \alpha u + \beta v + \gamma w \Leftrightarrow \begin{cases} \alpha + 3\beta = 1 \\ 2\alpha = 0 \\ 4\alpha + \beta + 2\gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \beta = \frac{1-\alpha}{3} = \frac{1}{3} \\ \alpha = 0 \\ \gamma = \frac{-4\alpha-\beta}{2} = -\frac{1}{6} \end{cases}$$

$$\begin{aligned}
(0, 1, 0) = \alpha u + \beta v + \gamma w &\Leftrightarrow \begin{cases} \alpha + 3\beta = 0 \\ 2\alpha = 1 \\ 4\alpha + \beta + 2\gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \beta = \frac{-\alpha}{3} = -\frac{1}{6} \\ \alpha = \frac{1}{2} \\ \gamma = \frac{-4\alpha - \beta}{2} = -\frac{11}{12} \end{cases} \\
(1, 0, 0) = \alpha u + \beta v + \gamma w &\Leftrightarrow \begin{cases} \alpha + 3\beta = 0 \\ 2\alpha = 0 \\ 4\alpha + \beta + 2\gamma = 1 \end{cases} \Leftrightarrow \begin{cases} \beta = \frac{-\alpha}{3} = 0 \\ \alpha = 0 \\ \gamma = \frac{1 - 4\alpha - \beta}{2} = \frac{1}{2} \end{cases}
\end{aligned}$$

Then,

$$\begin{aligned}
f(x, y, z) &= xf(1, 0, 0) + yf(0, 1, 0) + zf(0, 0, 1) \\
&= x \left[\frac{1}{3}f(v) - \frac{1}{6}f(w) \right] + y \left[\frac{1}{2}f(u) - \frac{1}{6}f(v) - \frac{11}{12}f(w) \right] + z \left[\frac{1}{2}f(w) \right] \\
&= \left(\frac{1}{2}y \right) f(u) + \left(\frac{1}{3}x - \frac{1}{6}y \right) f(v) - \left(\frac{1}{6}x + \frac{11}{12}y - \frac{1}{2}z \right) f(w) \\
&= \frac{1}{2}y \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \left(\frac{1}{3}x - \frac{1}{6}y \right) \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \left(\frac{1}{6}x + \frac{11}{12}y - \frac{1}{2}z \right) \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} \\
&= \left(\frac{x+4y}{3}, \frac{22x+13y-18z}{12}, \frac{-2x-53y+30z}{12} \right)
\end{aligned}$$

Exercise 2.6

Let $E = \mathbb{R}_n[X]$ the vector space of polynomials of degree less than or equal to n and $f : E \rightarrow E$ defined by :

$$f(P) = P + (1 - X)P'$$

1. Show that f is a linear application 2. Give a basis of $\ker f$ 3. Give a basis of $\text{Im } f$.

Solution

1. Show that f is a linear application

Let $P, Q \in \mathbb{R}_n[X]$, and $\lambda \in \mathbb{k} = \mathbb{R}$, we have

(i).

$$\begin{aligned}
f(P + Q) &= (P + Q) + (1 - X)(P + Q)' \\
&= (P + Q) + (1 - X)(P' + Q') \\
&= (P + (1 - X)P') + (Q + (1 - X)Q') = f(P) + f(Q).
\end{aligned}$$

(ii).

$$f(\lambda P) = (\lambda P) + (1 - X)(\lambda P)' = \lambda(P + (1 - X)P') = \lambda f(P),$$

from (i) and (ii), we deduce that f is a linear application.

2. give a basis of $\ker f$

$$\ker(f) = f^{-1}(\{0_{\mathbb{R}_n[X]}\}) = \{P \in \mathbb{R}_n[X] / f(P) = 0_{\mathbb{R}_n[X]}\}.$$

We solve the equation $f(P) = 0_{\mathbb{R}_n[X]}$.

$$\begin{aligned} f(P) = 0_{\mathbb{R}_n[X]} &\Leftrightarrow P + (1 - X)P' = 0 \Leftrightarrow \left(\sum_{i=0}^n a_i X^i\right) + (1 - X) \left(\sum_{i=1}^n i a_i X^{i-1}\right) = 0 \\ &\Leftrightarrow \left(\sum_{i=0}^n a_i X^i\right) + \left(\sum_{i=1}^n i a_i X^{i-1}\right) - \left(\sum_{i=1}^n i a_i X^i\right) = 0 \\ &\Leftrightarrow \sum_{i=0}^n a_i X^i + \sum_{i=0}^{n-1} (i+1) a_{i+1} X^i - \sum_{i=1}^n i a_i X^i = 0 \\ &\Leftrightarrow (a_0 + a_1) + \left(\sum_{i=1}^{n-1} ((1-i) a_i + (i+1) a_{i+1}) X^i\right) + a_n (1-n) X^n = 0 \\ &\Leftrightarrow \begin{cases} a_0 + a_1 = 0 \\ (i+1) a_{i+1} + (1-i) a_i = 0, \quad i \in \{1, \dots, n-1\} \\ a_n = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} a_0 = -a_1 \\ a_{i+1} = \frac{i-1}{i+1} a_i, \quad i \geq 1 \end{cases} \Leftrightarrow \begin{cases} a_0 = -a_1 \\ a_2 = \dots = a_n = 0 \end{cases} \end{aligned}$$

Then,

$$P(X) = \sum_{i=0}^n a_i X^i = a_0 (1 - X).$$

$$\ker f = \text{Span}(Q = 1 - X).$$

3. give a basis of $\text{Im } f$.

$$\begin{aligned}
 \text{Im}(f) &= \left\{ f(P)/P = \sum_{i=0}^n a_i X^i \in \mathbb{R}_n[X] \right\} \\
 &= \left\{ P + (1-X)P'/P = \sum_{i=0}^n a_i X^i \in \mathbb{R}_n[X] \right\} \\
 &= \left\{ (a_0 + a_1) + \left(\sum_{i=1}^{n-1} ((1-i)a_i + (i+1)a_{i+1}) X^i \right) + (a_n(1-n)X^n)/a_i \in \mathbb{R} \right\} \\
 &= \{ a_0 + a_1 + 2a_2X + \dots + ((2-n)a_{n-1} + (n)a_n)X^{n-1} + (a_n(1-n)X^n)/a_i \in \mathbb{R} \} \\
 &= \left\{ (a_0 + a_1) \times \underbrace{1}_{P_0} + a_2 \underbrace{(2X - X^2)}_{P_1} + \dots + a_n \underbrace{(nX^{n-1} + (1-n)X^n)}_{P_{n-1}} / a_i \in \mathbb{R} \right\}
 \end{aligned}$$

then,

$$\text{Im } f = \text{Span}(P_0 = 1, P_1 = 2X - X^2, P_2 = 3X^2 - 2X^3, \dots, P_n = nX^{n-1} + (1-n)X^n).$$

2.6 Terminology translation

English	Frensh
Linear application	Application linéaire
Kernel	Noyau
Rank theorem	Théorème du rang
Inverse image	Image réciproque
Direct image	Image directe

Matrices

3.1 Definitions

Definition 3.1

Let n, m be two strictly positive integers. A matrix $n \times m$ with coefficients in $\mathbb{k}$ is an array of elements of $\mathbb{k}$ with n rows and m columns, which we denote by

$$A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{pmatrix}$$

- The terms a_{ij} are called the coefficients of the matrix A .
- The matrix A is said of type (n, m) .
- The set of matrices with n rows and m columns with coefficients in $\mathbb{k}$ is denoted by $M_{n,m}(\mathbb{k})$.

Example 3.1

Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & 3 & 1 & 5 \end{pmatrix}$$

A is a 3-row, 4-column matrix with coefficients in $\mathbb{R}$, so $A \in M_{3,4}(\mathbb{R})$

The element a_{32} is in the third row and second column, so $a_{32} = 3$.

Definition 3.2

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ be a matrix of $M_{n,m}(\mathbb{k})$.

- If $n = m$, the matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is called a square matrix. The set of square matrices of order n is denoted by $M_n(\mathbb{K})$.

$$A = (a_{ij})_{1 \leq i, j \leq n} = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix}$$

- If $n = 1$, the matrix A is called a row matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \end{pmatrix}$$

- If $m = 1$, the matrix A is called a column matrix.

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{pmatrix}$$

Definition 3.3

A matrix $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$, whose all the elements are zero is called a zero matrix.

$$A = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} = (0).$$

Definition 3.4

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a square matrix. The elements a_{ii} are called the diagonal coefficients of A .

Definition 3.5

A matrix is said to be diagonal when it is square and its non-diagonal coefficients are zero.

It is of the form :

$$D = \begin{pmatrix} a_{11} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & a_{22} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & a_{nn} \end{pmatrix}$$

Definition 3.6

A matrix A is said to be upper triangular (respectively : lower) when it is square and its non-diagonal coefficients below (respectively : above) the diagonal are zero.

Upper triangular matrix : $a_{ij} = 0$ for all $i > j$

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ 0 & a_{22} & a_{23} & . & . & a_{2n} \\ 0 & 0 & a_{33} & . & . & a_{3n} \\ . & . & & . & . & . \\ . & . & & & . & . \\ 0 & 0 & . & . & 0 & a_{nn} \end{pmatrix}$$

Lower triangular matrix : $a_{ij} = 0$ for all $i < j$

$$A = \begin{pmatrix} a_{11} & 0 & . & . & . & 0 \\ a_{21} & a_{22} & 0 & . & . & 0 \\ a_{31} & a_{32} & a_{33} & 0 & . & 0 \\ . & . & & . & . & . \\ . & . & & & . & 0 \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

Definition 3.7

The identity matrix of size n is an $n \times n$ matrix with ones on the main diagonal and zeroes elsewhere. We denote it by I_n :

$$I_n = \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & & . & . & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & 0 & 1 \end{pmatrix}$$

3.2 Matrix associated with a linear application

In this section, E and F are finite dimensional vector spaces over a field $\mathbb{k}$, with dimensions m and n respectively. Let $B_E = (u_1, u_2, \dots, u_m)$ and $B_F = (v_1, v_2, \dots, v_n)$ bases of E and F respectively.

Definition 3.8

Let f be a linear application from E to F . The matrix of f in the bases B_E and B_F denoted $M_{B_E, B_F}(f)$ is the the matrix of the type (n, m) with coefficients in $\mathbb{k}$, whose elements of the

j^{-th} column are the coordinates of the vector $f(u_j)$ in the basis B_F .

$$M_{B_E, B_F}(f) = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}},$$

with

$$f(u_j) = \sum_{i=1}^n a_{ij} \times v_i.$$

Example 3.2

Let f be a linear application of $\mathbb{R}^2$ in $\mathbb{R}^3$ defined by

$$\forall (x, y) \in \mathbb{R}^2, f(x, y) = (x + y, 2x - 3y, x + 2y)$$

We consider the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

$$B_0 = \{e_1 = (1, 0), e_2 = (0, 1)\}$$

$$B_1 = \{e'_1 = (1, 0, 0), e'_2 = (0, 1, 0), e'_3 = (0, 0, 1)\}.$$

We have,

$$f(e_1) = f(1, 0) = (1, 2, 1) = e'_1 + 2e'_2 + e'_3,$$

so, the elements of the first column are 1, 2 and 1. Then

$$C_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$f(e_2) = f(0, 1) = (1, -3, 2) = e'_1 - 3e'_2 + 2e'_3,$$

so, the elements of the second column are 1, -3 and 2. Then

$$C_2 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Then, the matrix associated with the application f in the bases B_0 and B_1 is

$$M_{B_0, B_1}(f) = \begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 1 & 2 \end{pmatrix}.$$

Example 3.3

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear application defined by

$$\forall (x, y) \in \mathbb{R}^2, f(x, y) = (x + 3y, 2x - 5y)$$

Determine the matrix associated with f in the basics $B = \{u_1 = (1, 2), u_2 = (3, -1)\}$.

We have,

$$\begin{aligned} f(u_1) = \alpha u_1 + \beta u_2 &= \Leftrightarrow (7, -8) = (\alpha + 3\beta, 2\alpha - \beta) \\ &\Leftrightarrow \begin{cases} \alpha + 3\beta = 7 \\ 2\alpha - \beta = -8 \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = -\frac{17}{7} \\ \beta = \frac{22}{7} \end{cases} \end{aligned}$$

so, the elements of the first column are $-\frac{17}{7}$ and $\frac{22}{7}$. Then

$$C_1 = \begin{pmatrix} -\frac{17}{7} \\ \frac{22}{7} \end{pmatrix},$$

and

$$\begin{aligned} f(u_2) = \alpha u_1 + \beta u_2 &\Leftrightarrow (0, 11) = (\alpha + 3\beta, 2\alpha - \beta) \\ &\Leftrightarrow \begin{cases} \alpha + 3\beta = 0 \\ 2\alpha - \beta = 11 \end{cases} \\ &\Leftrightarrow \begin{cases} \alpha = \frac{33}{7} \\ \beta = -\frac{11}{7} \end{cases} \end{aligned}$$

so, the elements of the second column are $\frac{33}{7}$ and $-\frac{11}{7}$. Then

$$C_2 = \begin{pmatrix} \frac{33}{7} \\ -\frac{11}{7} \end{pmatrix}$$

Then, the matrix associated with the application f in the basis B

$$M_B(f) = \begin{pmatrix} -\frac{17}{7} & \frac{33}{7} \\ \frac{22}{7} & -\frac{11}{7} \end{pmatrix}.$$

Proposition 3.1

Let $f \in L_{\mathbb{K}}(E, F)$ and $A = M_{B_E, B_F}(f)$. If $u = \sum_{i=1}^m \alpha_i \times u_i$ and $f(u) = \sum_{i=1}^n \beta_i \times v_i$, then we have :

$$Y = AX, \quad \text{with} \quad X = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$$

Example 3.4

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear application whose matrix in the canonical basis is

$$A = M_B(f) = \begin{pmatrix} 1 & -3 \\ 5 & -4 \end{pmatrix}, \text{ with } B = \{e_1 = (1, 0), e_2 = (0, 1)\}.$$

So, for all real numbers x and y , we have :

$$\begin{pmatrix} 1 & -3 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 3y \\ 5x - 4y \end{pmatrix},$$

then,

$$f(x, y) = (x - 3y, 5x - 4y).$$

Proposition 3.2

Let $f, g \in L_{\mathbb{K}}(E, F)$. We have

$$M_{B_E, B_F}(f + g) = M_{B_E, B_F}(f) + M_{B_E, B_F}(g)$$

$$M_{B_E, B_F}(\lambda f) = \lambda \times M_{B_E, B_F}(f), \text{ where } \lambda \in \mathbb{K}.$$

Proposition 3.3

Let G be a finite dimensional vector space over a field $\mathbb{K}$, with dimension p . Let $B_G = (w_1, w_2, \dots, w_p)$ base of G . let $f \in L_{\mathbb{K}}(E, F)$ and $g \in L_{\mathbb{K}}(F, G)$.

We have

$$M_{B_E, B_G}(g \circ f) = M_{B_F, B_G}(g) \times M_{B_E, B_F}(f).$$

3.3 Equality of two matrices**Definition 3.9**

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ and $B = (b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$, be two matrices of $M_{n,m}(\mathbb{K})$. We say that $A = B$ if all the elements of A are equal to the corresponding elements of B .

$$A = B \Leftrightarrow a_{ij} = b_{ij}, \text{ for all } i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

Example 3.5

Let

$$A = \begin{pmatrix} 2 & x \\ y + 1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} x - y & 1 \\ x - 1 & 3 \end{pmatrix}$$

Let us determine x and y so that the two matrices are equal.

$$A = B \Leftrightarrow \begin{cases} x - y = 2 \\ x = 1 \\ y + 1 = x - 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}.$$

3.4 Transpose of a matrix

Definition 3.10

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ be a matrix of $M_{n,m}(\mathbb{K})$. We call the transpose of A the matrix $(a_{ji})_{\substack{1 \leq j \leq m \\ 1 \leq i \leq n}}$ of $M_{m,n}(\mathbb{K})$ and we denote ${}^tA$.

$$A = B \Leftrightarrow a_{ij} = b_{ij}, \text{ for all } i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

Example 3.6

Let

$$A = \begin{pmatrix} 1 & 5 \\ 2 & 0 \\ -3 & 4 \end{pmatrix}$$

Then, the transpose of A is the matrix defined by

$${}^tA = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 4 \end{pmatrix}.$$

Property 3.1

Let A and B be two matrices of $M_{n,m}(\mathbb{K})$. Let $\lambda \in \mathbb{K}$. Then, we have :

$${}^t({}^tA) = A, \quad {}^t(\lambda A) = \lambda({}^tA), \quad {}^t(A + B) = {}^tA + {}^tB.$$

Property 3.2

Let A be a matrix of $M_{n,m}(\mathbb{K})$ and B a matrix of $M_{m,p}(\mathbb{K})$. Then, we have :

$${}^t(A \times B) = ({}^tB) \times ({}^tA).$$

3.5 Trace of matrix

Definition 3.11

The sum of the diagonal elements of a square matrix A is called the trace of A , denoted by $Tr(A)$.

Example 3.7

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \\ 1 & -1 & 7 \end{pmatrix},$$

its trace is

$$\text{Tr}(A) = \sum_{i=1}^3 a_{ii} = a_{11} + a_{22} + a_{33} = (1) + (-4) + (7) = 4.$$

Theorem 3.1

Let A and B be two matrices of order n , then

1. $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$.
2. $\forall \lambda \in \mathbb{K}, \text{Tr}(\lambda A) = \lambda \text{Tr}(A)$.
3. $\text{Tr}({}^t A) = \text{Tr}(A)$.
4. $\text{Tr}(AB) = \text{Tr}(BA)$.

3.6 Operations on matrices**3.6.1 Multiplication par un scalaire**

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ be a matrix of $M_{n,m}(\mathbb{K})$ and $\lambda \in \mathbb{K}$. The product of A by λ is the matrix λA of $M_{n,m}(\mathbb{K})$ defined by :

$$\lambda A = (\lambda \times a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}.$$

Example 3.8

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 5 \end{pmatrix},$$

then,

$$2A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & -8 & 10 \end{pmatrix}, \quad -3A = \begin{pmatrix} -3 & -6 & -9 \\ 0 & 12 & -15 \end{pmatrix}.$$

3.6.2 Sum of two matrices

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$ and $B = (b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$, be two matrices of $M_{n,m}(\mathbb{K})$. The sum of matrices A and B is matrix $A + B$ of $M_{n,m}(\mathbb{K})$ defined by :

$$A + B = C = (c_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}, \text{ with } c_{ij} = a_{ij} + b_{ij}.$$

Example 3.9

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 1 & -4 \end{pmatrix},$$

then,

$$A + B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \\ 2 & -5 \end{pmatrix}, \quad \text{and} \quad 2A - 3B = \begin{pmatrix} 2 & 4 \\ 0 & -8 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} -9 & -3 \\ -6 & -15 \\ -3 & 12 \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ -6 & -23 \\ -1 & 10 \end{pmatrix}.$$

Property 3.3

Let A, B and C be three matrices of the same dimension, and α, β two scalars.

1. $A + B = B + A$, which characterizes the commutativity of matrix addition.
2. $A + (B + C) = (A + B) + C$, which characterizes the associativity of matrix addition.
3. $\alpha(A + B) = \alpha A + \beta B$.
4. $(\alpha + \beta)A = \alpha A + \beta A$.
5. $\alpha(\beta A) = (\alpha\beta)A$.

3.6.3 Product of two matrices

Let $A = (a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$, be a matrix of $M_{n,m}(\mathbb{K})$ and $B = (b_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq p}}$, be a matrix of $M_{m,p}(\mathbb{K})$. The product of matrices A and B is the matrix AB of $M_{n,p}(\mathbb{K})$ defined by

$$A \times B = (c_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}},$$

with,

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}.$$

Example 3.10

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -4 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix},$$

then,

$$A \times B = (c_{ij})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 2}}, \quad \text{with} \quad c_{ij} = \sum_{k=1}^2 a_{ik} b_{kj}.$$

$$c_{11} = \sum_{k=1}^2 a_{1k} b_{k1} = 1 \times 3 + 2 \times 2 = 7, \quad c_{12} = \sum_{k=1}^2 a_{1k} b_{k2} = 11,$$

$$c_{21} = \sum_{k=1}^2 a_{2k} b_{k1} = -8, \quad c_{22} = \sum_{k=1}^2 a_{2k} b_{k2} = -20,$$

$$c_{31} = \sum_{k=1}^2 a_{3k} b_{k1} = 1, \quad c_{32} = \sum_{k=1}^2 a_{3k} b_{k2} = -4,$$

so,

$$AB = \begin{pmatrix} 7 & 11 \\ -8 & -20 \\ 1 & -4 \end{pmatrix}.$$

Remark

The product of matrices is not commutative.

Example 3.11

Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

then we have :

$$AB = \begin{pmatrix} 8 & 3 \\ 9 & 5 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 2 & 3 \\ 3 & 11 \end{pmatrix}.$$

We can clearly see that the matrix product is not commutative. $AB \neq BA$.

Property 3.4

Let A, B , and C be three matrices. If the indicated operations exist, then we will assume the following equalities :

1. $A(B + C) = (AB) + (AC)$
2. $(A + B)C = (AC) + (BC)$
3. $A(BC) = (AB)C$.

Definition 3.12

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a square matrix of order n and k a non-zero natural number. We define the k -th power of A as follows :

$$A^k = \underbrace{A \times A \times A \times \dots \times A}_{k \text{ times}}$$

$$A^0 = I_n$$

Example 3.12

Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix},$$

then we have :

$$A^2 = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 21 \\ -7 & 22 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 1 & 21 \\ -7 & 22 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -19 & 108 \\ -36 & 89 \end{pmatrix}.$$

3.7 Inversion of a matrix**Definition 3.13**

Let $A = (a_{ij})_{1 \leq i, j \leq n}$, be a matrix of $M_n(\mathbb{K})$. We say that A is invertible if there exists a matrix B of $M_n(\mathbb{K})$ such that $AB = BA = I_n$. We denote $A^{-1} = B$.

Example 3.13

Let

$$A = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}.$$

We have,

$$A \times B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B \times A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

therefore, the matrix A is inverse invertible and $A^{-1} = B$.

Property 3.5

Let A be an invertible square matrix. Then,

$$({}^t A)^{-1} = {}^t (A^{-1}) \quad \text{and} \quad (A^{-1})^{-1} = A.$$

Proposition 3.1

Let A and B be two invertible square matrices of the same dimension. Then,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof

It suffices to show that $(B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I_n$. We have :

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}BI_n,$$

and

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Proposition 3.2

Let A be a square matrix of order two defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - cb \neq 0$.

Then, A is invertible and

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-cb} & -\frac{b}{ad-cb} \\ -\frac{c}{ad-cb} & \frac{a}{ad-cb} \end{pmatrix}$$

Proof

We verify that $AA^{-1} = A^{-1}A = I_2$.

Example 3.14

Let

$$A = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix},$$

the inverse of A is defined by

$$A^{-1} = \begin{pmatrix} \frac{2}{3 \times 2 - 1 \times 7} & -\frac{7}{3 \times 2 - 1 \times 7} \\ -\frac{1}{3 \times 2 - 1 \times 7} & \frac{3}{3 \times 2 - 1 \times 7} \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ 1 & -3 \end{pmatrix}.$$

3.7.1 Gauss's method for inverting a matrix

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix of $M_n(\mathbb{K})$. This method consists of performing elementary operations on the rows of the matrix A until it is transformed into the identity matrix I_n .

These elementary operations on the rows of the matrix A are :

$$R_i \leftarrow \lambda R_i \quad , \quad \lambda \neq 0$$

$$R_i \leftarrow R_i + \lambda R_j \quad , \quad \lambda \in \mathbb{k} \text{ and } j \neq i$$

So to apply this method, next to matrix A , we add the identity matrix to form the following table :

$$(A|I_n) : \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & . & . & a_{1n} & 1 & 0 & . & . & 0 \\ a_{21} & a_{22} & . & . & . & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} & 0 & 0 & . & . & 1 \end{array} \right)$$

On the rows of this matrix (called augmented matrix), we carry out elementary operations until we obtain the table

$$(I_n|B) : \left(\begin{array}{cccc|cccc} 1 & 0 & . & . & 0 & b_{11} & b_{12} & . & . & b_{1n} \\ 0 & 1 & . & . & . & b_{21} & b_{22} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & b_{n1} & . & . & . & b_{nn} \end{array} \right)$$

Then, $A^{-1} = B$.

Exemple 3.15

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Here is the augmented matrix

$$(A|I_3) : \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

By the following elementary operations $R_2 \leftarrow R_2 - 3R_1$ and $R_3 \leftarrow R_3 - 2R_1$, we find the following new augmented matrix

$$(A^{(1)}|B^{(1)}) : \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & -1 & -3 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array} \right) \begin{matrix} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \end{matrix}$$

We multiply the row $R_2^{(1)}$ by $(-\frac{1}{5})$, $(R_2^{(1)} \leftarrow (-\frac{1}{5}) R_2^{(1)})$, we find

$$(A^{(2)}|B^{(2)}) : \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} R_1^{(2)} \\ R_2^{(2)} \\ R_3^{(2)} \end{array}$$

By the following elementary operation $R_1^{(2)} \leftarrow R_1^{(2)} - 2R_2^{(2)}$, $R_3^{(2)} \leftarrow R_3^{(2)} + 3R_2^{(2)}$, we find

$$(A^{(3)}|B^{(3)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & -\frac{3}{5} & 1 \end{array} \right) \begin{array}{l} R_1^{(3)} \\ R_2^{(3)} \\ R_3^{(3)} \end{array}$$

We multiply the row $R_3^{(3)}$ by $(-\frac{5}{2})$ ($R_3^{(3)} \leftarrow -\frac{5}{2} R_3^{(3)}$), we find

$$(A^{(4)}|B^{(4)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & -\frac{5}{2} \end{array} \right) \begin{array}{l} R_1^{(4)} \\ R_2^{(4)} \\ R_3^{(4)} \end{array}$$

By the following elementary operations $R_2^{(4)} \leftarrow R_2^{(4)} - \frac{1}{5} R_3^{(4)}$ and $R_1^{(4)} \leftarrow R_1^{(4)} - \frac{3}{5} R_3^{(4)}$, we find

$$(A^{(5)}|B^{(5)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} & -\frac{5}{2} \end{array} \right) \begin{array}{l} R_1^{(5)} \\ R_2^{(5)} \\ R_3^{(5)} \end{array}$$

Therefore,

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & -\frac{5}{2} \end{pmatrix}.$$

3.8 Determinant

3.8.1 Determinant of a square matrix of order 2

Defenition 3.14

Let $A = (a_{ij})_{1 \leq i, j \leq 2}$ be a matrix of $M_2(\mathbb{K})$.

We call the determinant of A the number denoted $\det A$ or $|A|$

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}$$

Example 3.16

Let

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$$

we have,

$$\det A = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = (3) \times (4) - (1) \times (-2) = 16.$$

3.8.2 Determinant of a square matrix of order n

Definition 3.15

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a matrix of $M_n(\mathbb{K})$.

We define the determinant of A as follows :

$$\det A = \sum_{j=1}^n (-1)^{i+j} \times a_{ij} \times \det(A_{ij}),$$

where A_{ij} is the matrix obtained by removing the i -th row and the j -th column. Its determinant $\det(A_{ij})$ is called the minor of a_{ij} in A and the number $(-1)^{i+j} \det(A_{ij})$ is called the cofactor of a_{ij} in A .

Remark

To calculate $\det A$. We can expand along the j -th column

$$\det A = \sum_{i=1}^n (-1)^{i+j} \times a_{ij} \times \det(A_{ij}),$$

Example 3.17

Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & -5 & 7 \\ 6 & 1 & 2 \end{pmatrix}$$

We can expand along the rows or columns. Let's expand along the first row, we have,

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 2 & 4 \\ 3 & -5 & 7 \\ 6 & 1 & 2 \end{vmatrix} = \sum_{j=1}^3 (-1)^{1+j} \times a_{1j} \times \det(A_{1j}) \\ &= (1) \times \begin{vmatrix} -5 & 7 \\ 1 & 2 \end{vmatrix} - (2) \times \begin{vmatrix} 3 & 7 \\ 6 & 2 \end{vmatrix} + (4) \times \begin{vmatrix} 3 & -5 \\ 6 & 1 \end{vmatrix} \\ &= (-17) - 2(-36) + 4(33) = 187.\end{aligned}$$

We can check the result if we expand along the 2nd row or the 3rd row.

Let's expand along the 3rd column :

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 2 & 4 \\ 3 & -5 & 7 \\ 6 & 1 & 2 \end{vmatrix} = \sum_{i=1}^3 (-1)^{i+3} \times a_{i3} \times \det(A_{i3}) \\ &= (4) \times \begin{vmatrix} 3 & -5 \\ 6 & 1 \end{vmatrix} - (7) \times \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} + (2) \times \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} \\ &= 4(33) - 7(-11) + 2(-11) = 187.\end{aligned}$$

3.8.3 The properties of determinants

Property 3.6

Let A and B be two square matrices of order n . Then, we have :

1. $\det(AB) = \det A \det B$
2. $\det A = \det({}^t A)$
3. $\det A^{-1} = \frac{1}{\det A}$, In the case where A is invertible.

Property 3.7

The determinant of a matrix is zero if and only if the row vectors or column vectors are linked.

Property 3.8

If two rows or two columns of a determinant are swapped, the determinant changes sign while keeping the same absolute value.

Property 3.9

If we multiply a row (or column) of a matrix by a real number λ , the determinant of the new matrix is multiplied by λ .

Property 3.10

If we add a multiple of a row (or column) to a row (or column), the determinant does not change.

Theorem 3.2

A square matrix A is invertible if and only if $\det A \neq 0$.

Proposition 3.3

Let A be a square matrix of order n . We say that A is invertible if $\det A \neq 0$. And furthermore, we have :

$$A^{-1} = \frac{1}{\det A} {}^t C$$

where C is the cofactor matrix. $C = (c_{ij})_{1 \leq i, j \leq n}$, with $c_{ij} = (-1)^{i+j} |A_{ij}|$:

Example 3.18

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & 2 & -1 \end{pmatrix}$$

We have,

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & 2 & -1 \end{vmatrix}, \quad R_2 \leftarrow R_2 - R_1, R_3 \leftarrow R_3 - R_1 \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -7 \\ 1 & 1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -7 \\ 1 & -2 \end{vmatrix} = 5 \neq 0, \end{aligned}$$

so, the matrix A is invertible. and we have :

$$A^{-1} = \frac{1}{\det A} {}^t C.$$

We calculate the cofactors

$$\begin{aligned} c_{11} &= \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix} = 7 & c_{12} &= - \begin{vmatrix} 2 & -5 \\ 1 & -1 \end{vmatrix} = -3 & c_{13} &= \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1 \\ c_{21} &= - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3 & c_{22} &= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 & c_{23} &= - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1 \\ c_{31} &= \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -8 & c_{32} &= - \begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} = 7 & c_{33} &= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \end{aligned}$$

Then, the matrix C is

$$C = \begin{pmatrix} 7 & -3 & 1 \\ 3 & -2 & -1 \\ -8 & 7 & 1 \end{pmatrix},$$

therefore,

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 7 & 3 & -8 \\ -3 & -2 & 7 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} & \frac{3}{5} & -\frac{8}{5} \\ -\frac{3}{5} & -\frac{2}{5} & \frac{7}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

3.9 Transition matrix

Definition 3.16

Let E be a vector-space of dimension n , and $B = \{u_1, u_2, u_3, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_n\}$ two bases of E .

Each vector in the second basis has coordinates in the first, let p_{ij} denote the i -th coordinate of v_j in the basis B :

$$v_j = p_{1j}u_1 + p_{2j}u_2 + \dots + p_{nj}u_n.$$

The square matrix P with coefficients p_{ij} is called the transition matrix from B to B' . We denote it $P = M_{B',B}(Id_E)$.

Example 3.19

Let $B = \{u_1 = (1, 1, 1), u_2 = (1, 0, 2), u_3 = (-1, 3, 0)\}$

and $B' = \{v_1 = (2, 0, 0), v_2 = (1, -1, 1), v_3 = (1, 3, 0)\}$ two bases of $\mathbb{R}^3$.

We calculate the coordinates of u_i in the base B' .

$$u_1 = \alpha v_1 + \beta v_2 + \gamma v_3 \Leftrightarrow (2\alpha + \beta + \gamma, -\beta + 3\gamma, \beta) = (1, 1, 1)$$

$$\Leftrightarrow \begin{cases} 2\alpha + \beta + \gamma = 1 \\ -\beta + 3\gamma = 1 \\ \beta = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = -\frac{1}{3} \\ \gamma = \frac{2}{3} \\ \beta = 1 \end{cases}$$

$$u_2 = \alpha v_1 + \beta v_2 + \gamma v_3 \Leftrightarrow (2\alpha + \beta + \gamma, -\beta + 3\gamma, \beta) = (1, 0, 2)$$

$$\Leftrightarrow \begin{cases} 2\alpha + \beta + \gamma = 1 \\ -\beta + 3\gamma = 0 \\ \beta = 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = -\frac{1}{6} \\ \gamma = \frac{2}{3} \\ \beta = 2 \end{cases}$$

$$u_3 = \alpha v_1 + \beta v_2 + \gamma v_3 \Leftrightarrow (2\alpha + \beta + \gamma, -\beta + 3\gamma, \beta) = (-1, 3, 0)$$

$$\Leftrightarrow \begin{cases} 2\alpha + \beta + \gamma = -1 \\ -\beta + 3\gamma = 3 \\ \beta = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = -1 \\ \gamma = 1 \\ \beta = 0 \end{cases}$$

So,

$$\begin{aligned} u_1 &= \left(-\frac{1}{3}\right) \times v_1 + (1) \times v_2 + \left(\frac{2}{3}\right) \times v_3 \\ u_2 &= \left(-\frac{1}{6}\right) \times v_1 + (2) \times v_2 + \left(\frac{2}{3}\right) \times v_3 \\ u_3 &= (-1) \times v_1 + (0) \times v_2 + (1) \times v_3 \end{aligned}$$

Then, the transition matrix from B' to B is

$$P = M_{B,B'}(Id_E) = \begin{pmatrix} -\frac{1}{3} & -\frac{1}{6} & -1 \\ 1 & 2 & 0 \\ \frac{2}{3} & \frac{2}{3} & 1 \end{pmatrix}.$$

We can also find that

$$\begin{cases} v_1 = 12u_1 - 6u_2 - 4u_3 \\ v_2 = -3u_1 + 2u_2 + \frac{2}{3}u_3 \\ v_3 = 12u_1 - 6u_2 - 3u_3 \end{cases}$$

Then, the transition matrix from B to B' is

$$Q = M_{B',B}(Id_E) = \begin{pmatrix} 12 & -3 & 12 \\ -6 & 2 & -6 \\ -4 & \frac{2}{3} & -3 \end{pmatrix}.$$

We can verify that $Q = P^{-1}$.

Proposition 3.4

The transition matrix $P = M_{B',B}(Id_E)$ is invertible and its inverse is the transition matrix $P^{-1} = M_{B,B'}(Id_E)$.

Let $u \in E$ of coordinates $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in the basis B and of coordinates $(\beta_1, \beta_2, \dots, \beta_n)$ in

the basis B' .

We note $X = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ and $Y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$, then we have $Y = P^{-1}X$.

3.10 Change of base formula for a linear application

Let $f \in L(E, F)$, and let $B_E = \{u_1, u_2, \dots, u_n\}$ and $B'_E = \{u'_1, u'_2, \dots, u'_n\}$ two bases of E , and $B_F = \{v_1, v_2, \dots, v_n\}$ and $B'_F = \{v'_1, v'_2, \dots, v'_n\}$ two bases of F . We note $A = M_{B_E, B_F}(f)$, $B = M_{B'_E, B'_F}(f)$, $P = M_{B'_E, B_E}(Id_E)$ and $Q = M_{B_F, B'_F}(Id_F)$. Then, we have $B = Q^{-1}AP$.

$$\begin{array}{ccc} (E, B'_E) & \xrightarrow[B]{f} & (F, B'_F) \\ Id_E \downarrow P & & Q^{-1} \uparrow Id_F \\ (E, B_E) & \xrightarrow[A]{f} & (E, B_E) \end{array}$$

If $E = F$, we have

$$B = P^{-1}AP.$$

Definition 3.17

Two matrices A and B are called similar, if there is an invertible matrix P such that $A = P^{-1}BP$.

Example 3.20

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear application defined by

$$\forall (x, y, z) \in \mathbb{R}^3, f(x, y, z) = (x + y - z, 2x - y + 3z)$$

let $B_1 = \{u_1 = (1, 1, 1), u_2 = (1, 2, 0), u_3 = (1, 0, 0)\}$

and $B'_1 = \{u'_1 = (1, -1, 0), u'_2 = (0, 1, 0), u'_3 = (2, 1, -1)\}$ two bases of $\mathbb{R}^3$,

and $B_2 = \{v_1 = (1, 1), v_2 = (2, -1)\}$ and $B'_2 = \{v'_1 = (0, 1), v'_2 = (3, 1)\}$ two bases of $\mathbb{R}^2$.

We have :

$$f(u_1) = av_1 + bv_2 \Leftrightarrow (1, 4) = (a + 2b, a - b) \Leftrightarrow (a, b) = (3, -1),$$

and

$$f(u_2) = av_1 + bv_2 \Leftrightarrow (3, 0) = (a + 2b, a - b) \Leftrightarrow (a, b) = (1, 1),$$

and

$$f(u_3) = av_1 + bv_2 \Leftrightarrow (1, 2) = (a + 2b, a - b) \Leftrightarrow (a, b) = \left(\frac{5}{3}, -\frac{1}{3}\right),$$

Then, the matrix associated with the application f in the basis B_1 and B_2 is

$$A = M_{B_1, B_2}(f) = \begin{pmatrix} 3 & 1 & \frac{5}{3} \\ -1 & 1 & -\frac{1}{3} \end{pmatrix}$$

We calculate the transition matrix P from B_1 to B'_1

$$u'_1 = \alpha u_1 + \beta u_2 + \gamma u_3 \Leftrightarrow (1, -1, 0) = (\alpha + \beta + \gamma, \alpha + 2\beta, \alpha)$$

$$\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 1 \\ \alpha + 2\beta = -1 \\ \alpha = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma = \frac{3}{2} \\ \beta = -\frac{1}{2} \\ \alpha = 0 \end{cases}$$

and

$$u'_2 = \alpha u_1 + \beta u_2 + \gamma u_3 \Leftrightarrow (0, 1, 0) = (\alpha + \beta + \gamma, \alpha + 2\beta, \alpha)$$

$$\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha + 2\beta = 1 \\ \alpha = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma = -\frac{1}{2} \\ \beta = \frac{1}{2} \\ \alpha = 0 \end{cases}$$

and

$$u'_3 = \alpha u_1 + \beta u_2 + \gamma u_3 \Leftrightarrow (2, 1, -1) = (\alpha + \beta + \gamma, \alpha + 2\beta, \alpha)$$

$$\Leftrightarrow \begin{cases} \alpha + \beta + \gamma = 2 \\ \alpha + 2\beta = 1 \\ \alpha = -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma = 2 \\ \beta = 1 \\ \alpha = -1 \end{cases}$$

then, the matrix P is defined as

$$P = M_{B'_1, B_1}(Id) = \begin{pmatrix} 0 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{3}{2} & -\frac{1}{2} & 2 \end{pmatrix}$$

With the same method, we find the transition matrix Q^{-1} from B_2' to B_2

$$Q^{-1} = M_{B_2, B_2'}(Id) = \begin{pmatrix} \frac{2}{3} & -\frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

We apply the base change formula, we find

$$\begin{aligned} B = M_{B_1', B_2'}(f) = Q^{-1}AP &= \begin{pmatrix} \frac{2}{3} & -\frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 & 1 & \frac{5}{3} \\ -1 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{3}{2} & -\frac{1}{2} & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} & -\frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 & -\frac{1}{3} & \frac{4}{3} \\ -1 & \frac{2}{3} & \frac{4}{3} \end{pmatrix} \\ &= \begin{pmatrix} 3 & -\frac{4}{3} & -\frac{4}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix}. \end{aligned}$$

3.11 Rank of a matrix

Definition 3.18

The rank of a matrix A is the dimension of the vector subspace generated (spanned) by its columns or its rows.

Example 3.21

Let A be a matrix defined by :

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

By definition, we have

$$\begin{aligned} rk(A) &= \dim(\text{Span}(R_1 = (3, 1, 1), R_2 = (1, -1, 2))) \\ &\quad \text{or} \\ rk(A) &= \dim(\text{Span}(C_1 = (3, 1), C_2 = (1, -1), C_3 = (1, 2))) \end{aligned}$$

The vectors $R_1 = (3, 1, 1)$, $R_2 = (1, -1, 2)$ are linearly independent, then the rank of A is 2.

Proposition 3.5

The rank of a matrix with columns $C_1, C_2, \dots, C_n$ is not modified by the following three elementary operations on vectors :

1. A column can be multiplied by a non-zero scalar. $C_i \leftarrow \lambda C_i$ with $\lambda \neq 0$.
2. A multiple of another column C_j can be added to column C_i . $C_j \leftarrow C_j + \lambda C_i$ with $\lambda \in \mathbb{k}$ and $i \neq j$.
3. Two columns can be exchanged. $C_i \leftrightarrow C_j$.

3.12 Exercises

Exercise 3.1

Determine the representative matrices of linear applications relative to canonical bases

$$1. f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$(x, y, z) \mapsto (x + y - 2z, x - y + 3z)$$

$$2. f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$(x, y) \mapsto (2x + 3y, x - y, x + 4y)$$

$$3. f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$(x, y, z) \mapsto (x + z, x - 2z, x + y + z)$$

$$4. f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (2x + y, x + 3y)$$

Solution

Let $B = \{e_1 = (1, 0), e_2 = (0, 1)\}$ be the basis of $\mathbb{R}^2$ and $B' = \{e'_1 = (1, 0, 0), e'_2 = (0, 1, 0), e'_3 = (0, 0, 1)\}$ be the basis of $\mathbb{R}^3$.

$$1. f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad f(x, y, z) = (x + y - 2z, x - y + 3z)$$

We have,

$$f(e'_1) = (1, 1) = e_1 + e_2,$$

so, the elements of the first column are 1 and 1. Then

$$C_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f(e'_2) = (1, -1) = e_1 - e_2,$$

so, the elements of the second column are 1 and -1 . Then

$$C_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$f(e'_3) = (-2, 3) = -2e_1 + 3e_2,$$

so, the elements of the third column are -2 and 3 . Then

$$C_3 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Then, the matrix associated with the application f in the bases B and B' is

$$M_{B',B}(f) = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 3 \end{pmatrix}.$$

2. $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $f(x, y) = (2x + 3y, x - y, x + 4y)$

$$M_{B,B'}(f) = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 4 \end{pmatrix}.$$

3. $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(x, y, z) = (x + z, x - 2z, x + y + z)$

$$M_{B,B'}(f) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}.$$

4. $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $f(x, y) = (2x + y, x + 3y)$

$$M_{B,B'}(f) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Exercise 3.2

Consider the following matrices A , B and C , such that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & -2 \end{pmatrix}$$

Compute $2A - 3B$, $A + B$, $A^2 + B^2$, BC , AB , BA and ${}^tC^tB$.

Solution

1. Compute $2A - 3B$

$$2A - 3B = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} -3 & 3 \\ -6 & -9 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ -2 & -3 \end{pmatrix}$$

2. Compute $A + B$

$$A + B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}$$

3. Compute $A^2 + B^2$

$$\begin{aligned} A^2 + B^2 &= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 8 \\ 8 & 11 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 16 & 18 \end{pmatrix}. \end{aligned}$$

4. Compute BC, AB, BA

$$BC = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 3 \\ 8 & 17 & -4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 8 & 7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 8 & 13 \end{pmatrix}.$$

5. Compute ${}^t C {}^t B$

$${}^t C {}^t B = {}^t (BC) = \begin{pmatrix} -1 & 8 \\ 1 & 17 \\ 3 & -4 \end{pmatrix}.$$

Exercise 3.3

Calculate the determinants of the following matrices :

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 7 & 4 \\ -3 & 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ -4 & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 1 \\ 6 & 4 & 0 \\ 3 & 2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 5 & -4 \\ -4 & -5 & -17 & 12 \\ 4 & -1 & 1 & 3 \end{pmatrix}$$

Solution

1. Calculate the determinants of A

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 7 & 4 \\ -3 & 3 & 6 \end{pmatrix}$$

We develop along the first row, we obtain :

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 2 & -3 \\ 5 & 7 & 4 \\ -3 & 3 & 6 \end{vmatrix} = \sum_{j=1}^3 (-1)^{1+j} \times a_{1j} \times \det(A_{1j}) \\ &= (1) \times \begin{vmatrix} 7 & 4 \\ 3 & 6 \end{vmatrix} - (2) \times \begin{vmatrix} 5 & 4 \\ -3 & 6 \end{vmatrix} + (-3) \times \begin{vmatrix} 5 & 7 \\ -3 & 3 \end{vmatrix} = -162. \end{aligned}$$

2. Calculate the determinants of B

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ -4 & 2 & 1 \end{pmatrix}$$

We can do elementary operations to find more than zero in a row or column. For example, for the following operations, $C_2 \leftarrow C_2 - C_1$ and $C_3 \leftarrow C_3 + C_1$, we obtain

$$\begin{aligned} \det B &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ -4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 7 \\ -4 & 6 & -3 \end{vmatrix} \\ &= (1) \times \begin{vmatrix} 1 & 7 \\ 6 & -3 \end{vmatrix} = -45. \end{aligned}$$

3. Calculate the determinants of C

$$C = \begin{pmatrix} 2 & -1 & 1 \\ 6 & 4 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

By the operation $R_1 \leftarrow R_1 - R_2$, we get

$$\begin{aligned} \det C &= \begin{vmatrix} 2 & -1 & 1 \\ 6 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -3 & 0 \\ 6 & 4 & 0 \\ 3 & 2 & 1 \end{vmatrix} \\ &= (1) \times \begin{vmatrix} -1 & -3 \\ 6 & 4 \end{vmatrix} = 14. \end{aligned}$$

3. Calculate the determinants of D

$$D = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 5 & -4 \\ -4 & -5 & -17 & 12 \\ 4 & -1 & 1 & 3 \end{pmatrix}$$

We have, $R_3 = 2R_1 - 3R_2$, then

$$\det D = \begin{vmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 5 & -4 \\ -4 & -5 & -17 & 12 \\ 4 & -1 & 1 & 3 \end{vmatrix} = 0.$$

Exercise 3.4

We consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

1. Show that A is invertible.
2. Determine A^{-1} using the cofactors.

3. Determine A^{-1} using the Gauss-Jordan method.

Solution

1. Show that A is invertible.

We have :

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \sum_{j=1}^3 (-1)^{1+j} \times a_{1j} \times \det(A_{1j}) \\ &= (1) \times \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - (-1) \times \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + (1) \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 4 \neq 0,\end{aligned}$$

so, A is invertible.

2. Determine A^{-1} using the cofactors.

we have :

$$A^{-1} = \frac{1}{\det A} {}^t C.$$

We calculate the cofactors

$$\begin{aligned}c_{11} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 & c_{12} &= - \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -2 & c_{13} &= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \\ c_{21} &= - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 & c_{22} &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 & c_{23} &= - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2 \\ c_{31} &= \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 & c_{32} &= - \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2 & c_{33} &= \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3\end{aligned}$$

Then, the matrix C is

$$C = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix},$$

therefore,

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

3. Determine A^{-1} using the Gauss-Jordan method.

Here is the augmented matrix

$$(A|I_2) : \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

By the following elementary operations $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - R_1$, we find the following new augmented matrix

$$(A^{(1)}|B^{(1)}) : \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \end{array}$$

We multiply the row $R_2^{(1)}$ by $(\frac{1}{3})$, $(R_2^{(1)} \leftarrow (\frac{1}{3}) R_2^{(1)})$, we find

$$(A^{(2)}|B^{(2)}) : \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_1^{(2)} \\ R_2^{(2)} \\ R_3^{(2)} \end{array}$$

By the following elementary operation $R_1^{(2)} \leftarrow R_1^{(2)} + R_2^{(2)}$, $R_3^{(2)} \leftarrow R_3^{(2)} - 2R_2^{(2)}$, we find

$$(A^{(3)}|B^{(3)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{array} \right) \begin{array}{l} R_1^{(3)} \\ R_2^{(3)} \\ R_3^{(3)} \end{array}$$

We multiply the row $R_3^{(3)}$ by $(\frac{3}{4})$ ($R_3^{(3)} \leftarrow \frac{3}{4} R_3^{(3)}$), we find

$$(A^{(4)}|B^{(4)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array} \right) \begin{array}{l} R_1^{(4)} \\ R_2^{(4)} \\ R_3^{(4)} \end{array}$$

By the following elementary operations $R_2^{(4)} \leftarrow R_2^{(4)} + \frac{2}{3} R_3^{(4)}$ and $R_1^{(4)} \leftarrow R_1^{(4)} - \frac{1}{3} R_3^{(4)}$, we find

$$(A^{(5)}|B^{(5)}) : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array} \right) \begin{array}{l} R_1^{(5)} \\ R_2^{(5)} \\ R_3^{(5)} \end{array}$$

Therefore,

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

Exercise 3.5

Let A be a square matrix of order 2 defined by

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.$$

1. Compute A^2 .
2. Determine α and β such that $A^2 + \alpha A + \beta I_2 = 0$.

3. Deduce that A is invertible and determine A^{-1} .

Solution

1. Compute A^2 .

$$A^2 = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 20 & 8 \end{pmatrix}$$

2. Determine α and β such that $A^2 + \alpha A + \beta I_2 = 0$.

$$\begin{aligned} A^2 + \alpha A + \beta I_2 = 0 &\Leftrightarrow \begin{pmatrix} 13 & 5 \\ 20 & 8 \end{pmatrix} + \alpha \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} 13 + 3\alpha + \beta & 5 + \alpha \\ 20 + 4\alpha & 8 + 2\alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 13 + 3\alpha + \beta = 0 \\ 5 + \alpha = 0 \\ 20 + 4\alpha = 0 \\ 8 + 2\alpha + \beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = -5 \\ \beta = 2 \end{cases} \end{aligned}$$

3. Deduce that A is invertible

We have,

$$\det A = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 3 \times 2 - 4 \times 1 = 2 \neq 0,$$

then, A is invertible.

Determine A^{-1}

$$A^2 - 5A + 2I_2 = 0 \Leftrightarrow -\frac{1}{2}A^2 + \frac{5}{2}A = I_2$$

$$\Leftrightarrow A \left(-\frac{1}{2}A + \frac{5}{2}I_2 \right) = \left(-\frac{1}{2}A + \frac{5}{2}I_2 \right) A = I_2,$$

then, there exists a matrix $B = -\frac{1}{2}A + \frac{5}{2}I_2$, such that $AB = BA = I_2$, so A is invertible and

$$A^{-1} = -\frac{1}{2}A + \frac{5}{2}I_2 = -\frac{1}{2} \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}.$$

Exercise 3.6

Let f be the linear map defined by :

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$(x, y, z) \longmapsto (-x + y + z, x - y + z, x + y - z)$$

1. Write the matrix associated with f relative to the canonical basis B of $\mathbb{R}^3$.
2. Consider the family B' of vectors of $\mathbb{R}^3$,

$$B' = \{u_1(1, -1, 0), u_2(1, 0, -1), u_3(1, 1, 1)\}$$

- a- Show that B' is a basis of $\mathbb{R}^3$.
- b- Determine the transition matrix P from B to B' and calculate P^{-1} .
- c- Write the matrix associated with f in the basis B' .

Solution

1. Write the matrix associated with f relative to the canonical basis B of $\mathbb{R}^3$.

We consider the canonical bases of $\mathbb{R}^3$.

$$B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}.$$

We have,

$$f(e_1) = (-1, 1, 1) = -e_1 + e_2 + e_3,$$

so, the elements of the first column are $-1, 1$ and 1 . Then

$$C_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$f(e_2) = (1, -1, 1) = e_1 - e_2 + e_3,$$

so, the elements of the second column are $1, -1$ and 1 . Then

$$C_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$f(e_3) = (1, 1, -1) = e_1 + e_2 - e_3,$$

so, the elements of the second column are $1, 1$ and -1 . Then

$$C_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Then, the matrix associated with the application f in the basis B is

$$M_B(f) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

2. Consider the family B' of vectors of $\mathbb{R}^3$,

$$B' = \{u_1(1, -1, 0), u_2(1, 0, -1), u_3(1, 1, 1)\}$$

a- Show that B' is a basis of $\mathbb{R}^3$.

$$\det(u_1, u_2, u_3) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 3 \neq 0,$$

then, B' is a basis of $\mathbb{R}^3$.

b- Determine the transition matrix P from B to B'

We have,

$$\begin{cases} u_1(1, -1, 0) = (1) \times e_1 + (-1) \times e_2 + (0) \times e_3 \\ u_2(1, 0, -1) = (1) \times e_1 + (0) \times e_2 + (-1) \times e_3 \\ u_3(1, 1, 1) = (1) \times e_1 + (1) \times e_2 + (1) \times e_3 \end{cases}$$

then,

$$P = M_{B', B}(Id) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Calculate P^{-1}

$$\begin{cases} u_1 = e_1 - e_2 \\ u_2 = e_1 - e_3 \\ u_3 = e_1 + e_2 + e_3 \end{cases} \Leftrightarrow \begin{cases} e_1 = \frac{1}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3 \\ e_2 = -\frac{2}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3 \\ e_3 = \frac{1}{3}u_1 - \frac{2}{3}u_2 + \frac{1}{3}u_3 \end{cases}$$

then,

$$P^{-1} = M_{B, B'}(Id) = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

c- Write the matrix associated with f in the basis B' .

We have, $u_1(1, -1, 0), u_2(1, 0, -1), u_3(1, 1, 1)$

$$f(u_1) = (1, -1, 0) = (-2, 2, 0) = -2u_1,$$

so, the elements of the first column are $-2, 2$ and 2 . Then

$$C_1 = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$f(u_2) = (-2, 0, 2) = -2u_2,$$

so, the elements of the second column are 0, -2 and 0. Then

$$C_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

$$f(u_3) = (1, 1, 1) = u_3,$$

so, the elements of the second column are 0, 0 and 1. Then

$$C_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then, the matrix associated with the application f in the basis B' is

$$M_{B'}(f) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2nd method :

We apply the base change formula, we find

$$\begin{aligned} M_{B'}(f) = P^{-1} \times M_B(f) \times P &= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -2 & -2 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

3.13 Terminology translation

English	Frensh
Matrix	Matrice
Column	Colonne
Diagonal coefficients	Coefficients diagonaux
Lower triangular	Triangulaire inférieure
Sum	Somme
Transition matrix	Matrice de passage
Row	Ligne
Square matrix	Matrice carrée
Upper triangular	Triangulaire supérieure
Identity matrix	Matrice identité
Product	Produit

Solving systems of linear equations

4.1 System of Linear Equations

Definition 4.1

A linear system of n equations in m variables $x_1, x_2, \dots, x_m$ is a list of n linear equations, it is of the form :

$$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2m}x_m = b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{im}x_m = b_i \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nm}x_m = b_n \end{cases}$$

The numbers a_{ij} are the coefficients of the system.

We can write (S) in matrix form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}}_A \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{pmatrix}}_X = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}}_B$$

We call $A \in M_{n,m}(\mathbb{k})$ the coefficient matrix of the system. $B \in M_{n,1}(\mathbb{k})$ is the right-hand side vector.

The vector $X \in M_{m,1}(\mathbb{k})$ is a solution to the system if and only if

$$AX = B.$$

Example 4.1

Consider the following three linear systems :

$$(S_1) : \begin{cases} x + y + z = 3 \\ 2x - y + 5z = 6 \\ x + 3y - z = 3 \end{cases}, \quad (S_2) : \begin{cases} x + y + 2z = 1 \\ -3x - 3y + 6z = 2 \\ x - y - z = 4 \end{cases}, \quad (S_3) : \begin{cases} x + y = 1 \\ 2x + 2y = 3 \end{cases}$$

1. (S_1) is a system of three equations with three unknowns x, y and z . The matrix form of (S_1) is :

$$(S_1) : \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 3 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}}_B,$$

admits an infinity of solutions of the following form

$$X \in \{(3 - 2t, t, t), t \in \mathbb{R}\}$$

2. (S_2) is a system of three equations with three unknowns x, y and z . The matrix form of (S_2) is :

$$(S_2) : \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ -3 & -3 & 6 \\ 1 & -1 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_B,$$

admits a unique solution

$$X = \left(\frac{5}{2}, -\frac{3}{2}, 0\right).$$

3. (S_3) is a system of three equations with three unknowns x, y and z . The matrix form of (S_3) is :

$$(S_3) : \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_B,$$

does not admit solutions.

Definition 4.2

The augmented matrix associated with system (S) is defined by

$$(A/B) : \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} & b_1 \\ a_{21} & a_{22} & \dots & a_{2m} & b_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nm} & b_n \end{pmatrix}$$

4.2 Solving linear systems

4.2.1 Solving Linear Systems Using Gauss method

Whatever the values n and p of the system, its solutions can be determined using the Gaussian elimination method.

The principle is as follows :

Using successive linear combinations, we transform the initial system, which we take as is without changing the order of the equations, into an upper triangular system (the associated matrix is upper triangular), which is then solved starting with the last of the transformed equations.

Example 4.2

Let's solve the following system :

$$(S) : \begin{cases} x + y + z = 4 & \dots\dots (1) \\ 2x - y + 3z = 7 & \dots\dots (2) \\ -x + 2y + 2z = 5 & \dots\dots (3) \end{cases}$$

The matrix form of (S) is :

$$(S) : \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ -1 & 2 & 2 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}}_B$$

In the first step of the method, we eliminate the unknown x in equations $Eq2$ and $Eq3$. So we replace equation $(Eq2)$ by $(Eq2) - 2(Eq1)$ and $(Eq3)$ by $(Eq3) + (Eq1)$.

After this first step, we obtain the equivalent system

$$(S') : \begin{cases} x + y + z = 4 \\ -3y + z = -1 \quad \dots\dots (4) \\ 3y + 3z = 9 \quad \dots\dots (5) \end{cases}$$

In the second step, the second row acts as a pivot if y is present (otherwise, we swap (Eq.4) with (Eq.5)). To eliminate y from equation (Eq.5), we replace it with $(Eq.5) + (Eq.4)$. We then obtain the following equivalent upper triangular system :

$$(S'') : \begin{cases} x + y + z = 4 \\ -3y + z = -1 \\ 4z = 8 \quad \dots (6) \end{cases}.$$

We solve the system by "ascending" starting with the last equation.

Eq.6 gives $z = 2$.

Eq.4 gives $y = \frac{z+1}{3} = 1$.

Eq.1 gives $x = 4 - y - z = 1$.

Therefore, the solution to (S) is

$$X = (1, 1, 2).$$

Remark 4.1

We can solve a system of n equations with n unknowns by transforming the augmented matrix (A/B) to the new matrix as follows :

$$\left(\begin{array}{cccccc|c} 1 & 0 & . & . & . & 0 & \alpha_1 \\ 0 & 1 & 0 & . & . & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 & . & 0 & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & \alpha_n \end{array} \right)$$

$\underbrace{\hspace{10em}}_{I_n}$

We use the following elementary operations :

$$L_i \leftarrow \lambda L_i, \text{ with } \lambda \neq 0 \quad (\text{For find 1})$$

$$L_i \leftarrow L_i + \lambda L_j, \text{ with } i \neq j \quad (\text{For find 0})$$

Then the solution is

$$X = (x_1, x_2, \dots, x_n) = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

Example 4.3

Let's solve the following system :

$$(S) : \begin{cases} x + y + z + t = 1 \\ x - y + 2z - t = 2 \\ -x + y + 3z + 2t = 2 \\ 3x - 2y + z + 3t = -4 \end{cases}$$

The matrix form of (S) is :

$$(S) : \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \\ -1 & 1 & 3 & 2 \\ 3 & -2 & 1 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \\ -4 \end{pmatrix}}_B$$

Write the augmented matrix

$$(A/B) : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 & 2 \\ -1 & 1 & 3 & 2 & 2 \\ 3 & -2 & 1 & 3 & -4 \end{pmatrix}$$

Use row operations to transform the matrix into Reduced Row-Echelon Form (RREF) :

Step 1 : Eliminate the first column below the pivot.

$$L_2 \leftarrow L_2 - L_1, \quad L_3 \leftarrow L_3 + L_1, \quad L_4 \leftarrow L_4 - 3L_1,$$

$$(A/B)^{(1)} : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -2 & 1 \\ 0 & 2 & 4 & 3 & 3 \\ 0 & -5 & -2 & 0 & -7 \end{pmatrix}$$

Step 2 : Scale the second row.

$$L_2^{(1)} \leftarrow -\frac{1}{2}L_2^{(1)}$$

$$(A/B)^{(2)} : \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 2 & 4 & 3 & 3 \\ 0 & -5 & -2 & 0 & -7 \end{pmatrix}$$

Step 3 : Eliminate the second column.

$$L_1^{(2)} \leftarrow L_1^{(2)} - L_2^{(2)}, \quad L_3^{(2)} \leftarrow L_3^{(2)} - 2L_2^{(2)}, \quad L_4^{(2)} \leftarrow L_4^{(2)} + 5L_2^{(2)}$$

$$(A/B)^{(3)} : \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 5 & 1 & 4 \\ 0 & 0 & -\frac{9}{2} & 5 & -\frac{19}{2} \end{pmatrix}$$

Step 4 : Scale the third row.

$$L_3^{(3)} \leftarrow \frac{1}{5}L_3^{(3)}$$

$$(A/B)^{(4)} : \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & -\frac{9}{2} & 5 & -\frac{19}{2} \end{pmatrix}$$

Step 5 : Eliminate the third column.

$$L_1^{(4)} \leftarrow L_1^{(4)} - \frac{3}{2}L_3^{(4)}, \quad L_2^{(4)} \leftarrow L_2^{(4)} + \frac{1}{2}L_3^{(4)}, \quad L_4^{(4)} \leftarrow L_4^{(4)} + \frac{9}{2}L_3^{(4)}$$

$$(A/B)^{(5)} : \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{59}{10} & -\frac{59}{10} \end{pmatrix}$$

Step 6 : Scale the fourth row.

$$L_4^{(5)} \leftarrow \frac{10}{59}L_4^{(5)}$$

$$(A/B)^{(6)} : \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{11}{10} & -\frac{1}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Step 7 : Eliminate the fourth column.

$$L_1^{(6)} \leftarrow L_1^{(6)} + \frac{3}{10}L_4^{(6)}, \quad L_2^{(6)} \leftarrow L_2^{(6)} - \frac{11}{10}L_4^{(6)}, \quad L_3^{(6)} \leftarrow L_3^{(6)} - \frac{1}{5}L_4^{(6)}$$

$$(A/B)^{(7)} : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

then, the solution is

$$X = (0, 1, 1, -1).$$

4.2.2 Solving Linear Systems Using Cramer method

Let

$$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n \end{cases}$$

a system of linear equations with n equations and n unknowns. This system can also be written

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_X = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_B$$

We define the matrix A_j as follows

$$A_j = \begin{pmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & & a_{2,j-1} & b_2 & a_{2,j+1} & \dots & a_{2n} \\ \vdots & & & & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

$\uparrow$
 j^{th} column

In other words, A_j is the matrix obtained by replacing the j^{th} column of A by the second member B .

Theorem 4.1

Let

$$AX = B$$

a system of linear equations with n equations and n unknowns. Suppose that $\det A \neq 0$. Then, the unique solution of the system is given by

$$x_j = \frac{\det(A_j)}{\det(A)}$$

Proof

We assumed that

$$\det A \neq 0.$$

Then, A is invertible. So

$$X = A^{-1}B,$$

the unique solution of the system. On the other hand, we have seen that

$$A^{-1} = \frac{1}{\det A} {}^t C,$$

where C is the cofactor matrix. $C = (c_{ij})_{1 \leq i, j \leq n}$ with $c_{ij} = (-1)^{i+j} |A_{ij}|$.

Then,

$$X = \frac{1}{\det A} {}^t C B$$

which gives

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} c_{11} & c_{21} & \cdot & \cdot & c_{n1} \\ c_{12} & c_{22} & \cdot & \cdot & c_{n2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ c_{1n} & c_{2n} & \cdot & \cdot & c_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

that's to say

$$x_1 = \frac{1}{\det A} (c_{11}b_1 + c_{21}b_2 + \dots + c_{n1}b_n)$$

$$x_i = \frac{1}{\det A} (c_{1i}b_1 + c_{2i}b_2 + \dots + c_{ni}b_n)$$

$$x_n = \frac{1}{\det A} (c_{1n}b_1 + c_{2n}b_2 + \dots + c_{nn}b_n)$$

But

$$c_{1i}b_1 + c_{2i}b_2 + \dots + c_{ni}b_n,$$

is the cofactor expansion of $\det(A_j)$ with respect to its i^{th} column. Therefore

$$x_j = \frac{\det(A_j)}{\det(A)}.$$

Example 4.4

Let's solve the following system :

$$(S) : \begin{cases} x - y - z = 0 \\ 2x + 3y + z = -1 \\ 3x - y - 4z = -7 \end{cases}$$

The matrix form of (S) is :

$$(S) : \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -4 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 0 \\ -1 \\ -7 \end{pmatrix}}_B,$$

let's calculate $\det(A)$,

$$\det(A) = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -4 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ -1 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0,$$

then, the system admits a unique solution.

We can apply the Cramer method, we have :

$$x = \frac{\det A_1}{\det A}, \quad y = \frac{\det A_2}{\det A}, \quad z = \frac{\det A_3}{\det A},$$

with,

$$A_1 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 3 & 1 \\ -7 & -1 & -4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 3 & -7 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & -7 \end{pmatrix},$$

then, the solution is

$$x = \frac{\det A_1}{\det A} = \frac{(-11)}{(-11)} = 1$$

$$y = \frac{\det A_2}{\det A} = \frac{22}{-11} = -2$$

$$z = \frac{\det A_3}{\det A} = \frac{-33}{-11} = 3$$

4.2.3 Resolution by the inverse matrix method of the coefficients

Let the following system :

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_A \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}}_X = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}}_B,$$

where, $A \in M_n(\mathbb{K})$ is an invertible square matrix. Then, the unique solution of the system is given by

$$X = A^{-1}B.$$

Example 4.5

Let's solve the following system :

$$(S) : \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 2 \\ 3 & -1 & -5 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}}_B,$$

The determinant of matrix A is -14 , therefore matrix A is invertible and its inverse is,

$$A^{-1} = \begin{pmatrix} \frac{13}{14} & \frac{2}{7} & -\frac{1}{14} \\ -\frac{11}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{5}{7} & \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

Then, the unique solution of the system is given by

$$X = A^{-1}B = \begin{pmatrix} \frac{13}{14} & \frac{2}{7} & -\frac{1}{14} \\ -\frac{11}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{5}{7} & \frac{1}{7} & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{18}{7} \\ -\frac{5}{7} \\ \frac{2}{7} \end{pmatrix}.$$

4.3 Number of solutions

Let (S) be a system of n linear equations with p unknowns. Therefore, (S) is equivalent to :

$$AX = B,$$

where $A \in M_{n,m}(\mathbb{K})$, $B \in M_{n,1}(\mathbb{K})$ and $X \in M_{m,1}(\mathbb{K})$.

Theorem 4.2

A linear system has either an infinite number of solutions, a unique solution, or no solution.

To understand this result, note that using Gauss's algorithm, we simply need to reason on an echelon system. There are several cases :

- If there exists a line of the form $0x_1 + 0x_2 + \dots + 0x_n = b$ with $b \neq 0$, then the system is incompatible.
- Otherwise, the system is compatible, more precisely, it has a unique solution if there is no secondary unknown, and it has an infinite number of solutions if there is at least one secondary unknown.

Proposition 4.1

A linear system has either

- no solution ($r < n$ and incompatible system).

- a unique solution ($r = m \leq n$ and compatible system).
- an infinite number of solutions ($r < m$ and compatible system).

Where, r is the rank of the matrix A , (augmented matrix).

4.3.1 Case where $n = m$ (A is a square matrix)

Homogeneous system ($B = 0$)

The system (S) is said to be homogeneous if the right-hand side is zero ($B = 0$).

The set of X such that $AX = 0$ constitutes the kernel of the application associated with A .

The kernel always contains the zero vector, but it can also contain non-zero vectors.

This type of system therefore has at least one solution, the zero solution.

- If A is invertible ($\det A \neq 0$)

The system has the unique solution $X = 0$, zero vector.

- If A is not invertible ($\det A = 0$)

The system has an infinity of solutions (in addition to the zero solution).

Example 4.6

- The system :

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 4 & 7 & -3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_B,$$

has the unique solution $X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, the determinant of the matrix is -3 , the rank of the matrix is 3.

- The system :

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & -4 & -3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_B,$$

has an infinity of solutions $X = \begin{pmatrix} -z \\ -z \\ -z \end{pmatrix} = -z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, with $z \in \mathbb{R}$. The determinant of the matrix is 0, the rank of the matrix is 2.

The set of these solutions constitutes a vector space of dimension 1 generated by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Non-homogeneous system ($B \neq 0$)

- **If A is invertible** ($\det A \neq 0$)

The system has the unique solution $X = A^{-1}B$.

- **If A is not invertible** ($\det A = 0$)

For there to be at least one solution, the rank of A must be the same as the rank of matrix $C = (A/B)$, the matrix formed by A to which B is attached.

If X_0 is a particular solution of the inhomogeneous system $AX = B$, then we have :

$$AX_0 = B,$$

then,

$$A(X - X_0) = 0.$$

Let $X = X - X_0$ be the solution of the homogeneous equation.

Therefore, the solution of the inhomogeneous system is $X = X + X_0$.

Example 4.7

- We consider the following system :

$$\underbrace{\begin{pmatrix} 1 & 1 & -2 & 3 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 6 & 7 \\ -1 & 4 & 2 & 5 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 10 \\ 5 \\ -8 \\ 3 \end{pmatrix}}_B,$$

the determinant of the matrix is 36, so the matrix A is invertible and its inverse is

$$A^{-1} = \begin{pmatrix} -\frac{19}{18} & \frac{8}{3} & -\frac{13}{18} & \frac{10}{9} \\ -\frac{11}{9} & \frac{3}{6} & -\frac{8}{9} & \frac{13}{9} \\ -\frac{11}{36} & \frac{1}{6} & \frac{1}{36} & \frac{1}{9} \\ \frac{8}{9} & -\frac{5}{3} & \frac{5}{9} & -\frac{7}{9} \end{pmatrix}.$$

Then the system has the unique solution $X = \begin{pmatrix} \frac{107}{9} \\ \frac{113}{9} \\ -\frac{19}{9} \\ -\frac{56}{9} \end{pmatrix}$.

- The system :

$$\underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 1 & -1 & 7 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_B,$$

has no solution. We can verify that the determinant of the matrix is 0.

The rank of the matrix

$$C = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 4 & 2 \\ 1 & -1 & 7 & 3 \end{pmatrix},$$

is 3 and the rank of the matrix A is 2.

- The system :

$$\underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 1 & -1 & 7 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_B,$$

has an infinity of solutions. We can verify that the determinant of the matrix is 0.

The rank of the matrix

$$C = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 4 & 2 \\ 1 & -1 & 7 & 1 \end{pmatrix},$$

is 2 and the rank of the matrix A is 2.

4.3.2 Case where the number of equations is different from the number of unknowns ($n \neq m$)

$n > m$ there are more equations than unknowns

Example 4.8

- The system :

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 5 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 5 \\ 4 \\ 3 \\ 1 \end{pmatrix}}_B,$$

has no solution. To verify this, either with the Gauss method, which will specify the impossibilities, either determine the rank of C and we compare it to that of A .

The rank of the matrix

$$C = \begin{pmatrix} 1 & 2 & 1 & 5 \\ -1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & -2 & 5 & 1 \end{pmatrix},$$

is 4 (because $\det C = 12 \neq 0$) and the rank of the matrix A is 3.

$n < m$ **there are fewer equations than unknowns**

Example 4.9

- The system :

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 1 \\ 3 & 2 & -4 & 5 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}}_B,$$

has an infinity solutions $X = \left(\frac{1-12t}{11}, \frac{19-5t}{22}, \frac{37+7t}{22}, t\right), t \in \mathbb{R}$.

4.4 Exercises

Exercise 4.1

Let the system :

$$(S) : \begin{cases} x + y - z = -4 \\ 2x + 3y + z = -1 \\ -x + y + 2z = 3 \end{cases}$$

1. Write the system (S) in the matrix form $AX = B$ and find the inverse of A .
2. Deduce the solution of the system (S) .

Solution

1. Write the system (S) in the matrix form $AX = B$

$$(S) \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}}_B$$

So,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

- The inverse of A

We have :

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{vmatrix}, & \begin{array}{l} C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 + C_1 \end{array} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \neq 0,\end{aligned}$$

then, the matrix A is invertible, and we have :

$$A^{-1} = \frac{1}{\det A} {}^t C.$$

We calculate the cofactors

$$\begin{aligned}c_{11} &= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5 & c_{12} &= - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = -5 & c_{13} &= \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5 \\ c_{21} &= - \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -3 & c_{22} &= \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 & c_{23} &= - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2 \\ c_{31} &= \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 & c_{32} &= - \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3 & c_{33} &= \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1\end{aligned}$$

Then, the matrix C is

$$C = \begin{pmatrix} 5 & -5 & 5 \\ -3 & 1 & -2 \\ 4 & -3 & 1 \end{pmatrix},$$

therefore

$$A^{-1} = \frac{1}{-5} \begin{pmatrix} 5 & -3 & 4 \\ -5 & 1 & -3 \\ 5 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{3}{5} & -\frac{4}{5} \\ 1 & -\frac{1}{5} & \frac{3}{5} \\ -1 & \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

2. Deduce the solution of the system (S)

$$AX = B \Leftrightarrow X = A^{-1}B = \begin{pmatrix} -1 & \frac{3}{5} & -\frac{4}{5} \\ 1 & -\frac{1}{5} & \frac{3}{5} \\ -1 & \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

Then, the solution of (S) is

$$X = (1, -2, 3).$$

Exercise 4.2

Solve the system by using Gaussian elimination.

$$(S_1) : \begin{cases} x + 2y - 2z = 3 \\ x - y + z = 6 \\ x + 3y - 3z = 2 \end{cases}, \quad (S_2) : \begin{cases} x + 2y - 2z = 3 \\ x - y + z = 6 \\ x + 3y - 3z = 1 \end{cases}$$

Solution

1.

$$(S_1) : \begin{cases} x + 2y - 2z = 3 & \dots (1) \\ x - y + z = 6 & \dots (2) \\ x + 3y - 3z = 2 & \dots (3) \end{cases},$$

In the first step, we eliminate the unknown x in equations $Eq2$ and $Eq3$. So we replace equation $(Eq2)$ by $(Eq2) - (Eq1)$ and $(Eq3)$ by $(Eq3) - (Eq1)$.

After this first step, we obtain the equivalent system

$$(S') : \begin{cases} x + y + z = 4 \\ -3y + 3z = 3 & \dots (4) \\ y - z = -1 & \dots (5) \end{cases}$$

In the second step, we eliminate y from equation $(Eq.5)$, we replace it with $(Eq.5) + \frac{1}{3}(Eq.4)$.

We then obtain the following equivalent upper triangular system :

$$(S'') : \begin{cases} x + y + z = 4 \\ -3y + 3z = 3 \\ 0 \times z = 0 & \dots (6) \end{cases}.$$

We now have an echelon form system in which $Eq.6$, $0 \times z = 0$, is satisfied by any number z .

Therefore, we have infinitely many solutions. Let $z = t$, where t is any number. $Eq.4$ implies $y = z - 1 = t - 1$. Finally, we get x from $Eq.1$, $x = 4 - y - z = 4 - (t - 1) - t = 5 - 2t$.

Infinitely many solutions are given by

$$x = 5 - 2t, y = -1 + t, z = t.$$

2. Note that the system of equations given here is the same as that in part (1) except for the right side of $Eq.3$. The same elementary operations performed in the solution to (S_1) yield the following echelon form for the system.

$$\begin{cases} x + y + z = 4 \\ -3y + 3z = 3 \\ 0 \times z = -1 \end{cases}$$

Since no number z satisfies the equation $0 \times z = -1$, the system has no solution.

Exercise 4.3

Determine, according to the value of the parameter $m \in \mathbb{R}$ and using the Gauss algorithm, the set of solutions of the system :

$$(S) : \begin{cases} x + y - z = 1 \\ 3x + y - z = 1 \\ x - 2y + 2z = m \end{cases}$$

Solution

We have

$$\begin{aligned} (S) : \begin{cases} x + y - z = 1 \\ 3x + y - z = 1 \\ x - 2y + 2z = m \end{cases} &\Leftrightarrow \begin{cases} x + y - z = 1 \\ -2y + 2z = -2 \\ -3y + 3z = m - 1 \end{cases} \quad \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array} \\ &\Leftrightarrow \begin{cases} x + y - z = 1 \\ y - z = 1 \\ -3y + 3z = m - 1 \end{cases} \quad R_2 \leftarrow -\frac{1}{2}R_2 \\ &\Leftrightarrow \begin{cases} x + y - z = 1 \\ y - z = 1 \\ 0 = m + 2 \end{cases} \quad R_3 \leftarrow R_3 + 3R_2 \end{aligned}$$

We can then discuss according to the values of m :

- If $m + 2 \neq 0$, that is, if $m \neq -2$, the system has no solutions.
- If $m = -2$, then the system is equivalent to

$$\begin{cases} x = 0 \\ y = 1 + z \end{cases},$$

then, the set of solutions is

$$\{(0, 1 + z, z), z \in \mathbb{R}\}.$$

Exercise 4.4

Determine the rank of the following systems and solve them.

$$(S_1) : \begin{cases} 3x - 2y = 7 \\ x + 3y = 4 \end{cases}, \quad (S_2) : \begin{cases} x - y + 2z = -3 \\ x + y - 3z = 6 \\ 2x + 3y + 5z = 3 \end{cases}, \quad (S_3) : \begin{cases} x + y - 2z = 3 \\ 3x + 3y - 6z = 10 \\ -2x - 4y + 3z = -7 \end{cases}$$

Solution

1.

$$(S_1) : \begin{cases} 3x - 2y = 7 \\ x + 3y = 4 \end{cases}$$

The matrix form of (S_1) is :

$$(S_1) : \underbrace{\begin{pmatrix} 3 & -2 \\ 1 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 7 \\ 4 \end{pmatrix}}_B,$$

we have,

$$\det A = \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = 11 \neq 0,$$

then, $rk(S_1) = rk(A) = 2$. Then the system (S_1) has an unique solution.

By the method of Cramer, we obtain

$$x = \frac{\begin{vmatrix} 7 & -2 \\ 4 & 3 \end{vmatrix}}{\det A} = \frac{29}{11} \quad \text{and} \quad y = \frac{\begin{vmatrix} 3 & 7 \\ 1 & 4 \end{vmatrix}}{\det A} = \frac{5}{11},$$

so, the set of solution of (S_1) is

$$\left\{ \left(\frac{29}{11}, \frac{5}{11} \right) \right\}.$$

2.

$$(S_2) : \begin{cases} x - y + 2z = -3 \\ x + y - 3z = 6 \\ 2x + 3y + 5z = 3 \end{cases}$$

The matrix form of (S_2) is :

$$(S_2) : \underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & -3 \\ 2 & 3 & 5 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}}_B,$$

we have,

$$\det A = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -3 \\ 2 & 3 & 5 \end{vmatrix} = 27 \neq 0,$$

then, $rk(S_2) = rk(A) = 3$. Then the system (S_2) has an unique solution.

By the method of Cramer, we obtain

$$x = \frac{\begin{vmatrix} -3 & -1 & 2 \\ 6 & 1 & -3 \\ 3 & 3 & 5 \end{vmatrix}}{\det A} = \frac{27}{27} = 1, \quad y = \frac{\begin{vmatrix} 1 & -3 & 2 \\ 1 & 6 & -3 \\ 2 & 3 & 5 \end{vmatrix}}{\det A} = \frac{54}{27} = 2, \quad \text{and} \quad z = \frac{\begin{vmatrix} 1 & -1 & -3 \\ 1 & 1 & 6 \\ 2 & 3 & 3 \end{vmatrix}}{\det A} = \frac{-27}{27} = -1$$

so, the set of solution of (S_2) is

$$\{(1, 2, -1)\}.$$

3.

$$(S_3) : \begin{cases} x + y - 2z = 3 \\ 3x + 3y - 6z = 10 \\ -2x - 4y + 3z = -7 \end{cases}$$

The matrix form of (S_3) is :

$$(S_3) : \underbrace{\begin{pmatrix} 1 & 1 & -2 \\ 3 & 3 & -6 \\ -2 & -4 & 3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 3 \\ 10 \\ -7 \end{pmatrix}}_B,$$

we have,

$$\det A = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 3 & -6 \\ -2 & -4 & 3 \end{vmatrix} = 0, \quad \text{because } R_2 = 3R_1.$$

then, $rk(S_2) = rk(A) \leq 2$. We choose a square submatrix of order 2, for example we take the following matrix

$$A' = \begin{pmatrix} 3 & 3 \\ -2 & -4 \end{pmatrix}$$

we have,

$$\det A = \begin{vmatrix} 3 & 3 \\ -2 & -4 \end{vmatrix} = -6 \neq 0,$$

then, $rk(S_3) = rk(A) = 2$.

we choose the following subsystem

$$(S'_3) : \begin{cases} 3x + 3y - 6z = 10 + 6z \\ -2x - 4y + 3z = -7 - 3z \end{cases}$$

The matrix form of (S'_3) is :

$$(S'_3) : \underbrace{\begin{pmatrix} 3 & 3 \\ -2 & -4 \end{pmatrix}}_{A'} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X'} = \underbrace{\begin{pmatrix} 10 + 6z \\ -7 - 3z \end{pmatrix}}_{B'},$$

Then the system (S'_3) has an unique solution. By the method of Cramer, we obtain

$$x = \frac{\begin{vmatrix} 10+6z & 3 \\ -7-3z & -4 \end{vmatrix}}{\det A'} = \frac{19+15z}{6}, \quad y = \frac{\begin{vmatrix} 3 & 10+6z \\ -2 & -7-3z \end{vmatrix}}{\det A'} = \frac{1-3z}{6},$$

we replace x and y in the equation (1), we obtain

$$x + y - 2z = 3 \Leftrightarrow \frac{19+15z}{6} + \frac{1-3z}{6} - 2z = 3 \Leftrightarrow \frac{19+15z+1-3z-12z}{6} = 3 \Leftrightarrow \frac{20}{6} = 3, \text{ (impossible)}$$

so, the system (S_3) has not a solution.

4.5 Terminology translation

English	Frensh
Linear équation	Equation linéaire
Unknown	Inconnue
Elementary operation	Opération élémentaire
Unique solution	Solution unique
Homogeneous system	Système homogène
Matrix form	Forme matricielle
Augmented matrix	Matrice augmentée
Cofactor	Cofacteur
Linear system	Système linéaire
Zero vector	Vecteur nul

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