Mathematical Analysis 3
Course Notes

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Contents

Preface

This handout is intended for second-year undergraduate students in the Mathematics program. It provides a comprehensive and in-depth exploration of the topics typically covered in a third course on mathematical analysis at the university level.

Designed for students of mathematics, physics, and engineering, this compendium bridges the gap between the foundational principles of calculus and the more abstract realms of advanced analysis. The material presented here is essential for developing a rigorous understanding of infinite processes, function spaces, and the analytical tools that underpin modern science and technology.

The journey through this text is organized into six logically structured chapters, each building upon the previous to form a coherent and powerful analytical framework.

Chapter 1 – Infinite Series. Our exploration begins with the concept of an infinite sum. Moving beyond intuition, we establish a rigorous theory of convergence and divergence. The chapter classifies various types of series—geometric, harmonic, and those with positive, alternating, or general terms—and introduces a rich collection of convergence tests: comparison, ratio, root, integral, and Raabe–Duhamel. The discussion culminates in the study of conditionally convergent series and the algebra of series, including the Cauchy product.

Chapter 2 – Sequences and Series of Functions. This chapter marks a transition from numbers to functions. We examine how a sequence or series of functions can converge to a limit function, distinguishing between pointwise and uniform convergence. The emphasis lies in understanding how uniform convergence preserves key properties such as continuity, integrability, and differentiability. Foundational results—such as Dini's theorem and the Weierstrass M-test—are presented, equipping the reader with both theoretical insight and practical techniques.

Chapter 3 – Power Series. Here, we study one of the most important classes of function series—those that behave like infinite polynomials. We introduce the concept of the radius of convergence and analyze the remarkable properties of power series within their interval of convergence, including term-by-term differentiation and integration. The chapter highlights the deep connection between power series and analytic functions, with special attention to Taylor and Maclaurin expansions—indispensable tools for approximation and differential equations.

Chapter 4 – Fourier Series. This chapter introduces an alternative yet equally powerful representation of functions—through trigonometric expansions. We explore how periodic functions can be decomposed into sums of sine and cosine terms. Topics include the computation of Fourier coefficients, convergence theorems (both pointwise and uniform), and the treatment of even and odd functions. Parseval's identity is derived, revealing the profound link between energy in the time domain and the sum of squared coefficients, with major applications in signal processing, physics, and engineering.

Chapter 5 – Improper Integrals. Extending the classical Riemann integral, we consider integrals over unbounded intervals or with unbounded integrands. A full theory of convergence is developed, parallel to that of Infinite Series, including comparison tests and the integral test. The subtle distinction between absolute and conditional convergence reappears, providing a unified understanding of infinite processes in both discrete and continuous contexts.

Chapter 6 – Integrals Depending on a Parameter. The final chapter investigates families of integrals defined by a variable parameter. We study the continuity, differentiability, and integrability of these functions with respect to the parameter, establishing the theorems that justify the interchange of limits, derivatives, and integrals. As a significant application, we present a detailed analysis of the Gamma function—a cornerstone of mathematical analysis that generalizes the factorial function.

Throughout this text, theoretical exposition is reinforced by a wealth of fully worked examples and carefully curated exercises at the end of each chapter. These aim not only to test comprehension but also to foster a deep and lasting mastery of the methods and ideas presented.

It is our sincere hope that this volume will serve as a clear guide, a reliable reference, and a source of intellectual inspiration—equipping students with the analytical sophistication required for advanced study in mathematics, physics, and related disciplines. The journey from discrete series to the continuous structures of functional analysis represents one of the most elegant transitions in modern mathematics, and this text is designed to accompany the reader every step of the way.

Chapter 1

Infinite Series

This chapter covers the generalities and definitions of infinite series. A infinite series is an expression of the form $\sum_{n=1}^{+\infty} u_n$, where u_n is called the n-th term of the series. To analyze such a series, we associate it with a sequence of partial sums (S_n), where S_n is the sum of the first n terms.

The nature of the series is then defined by the limit of this sequence:

- We say the series **converges** if the limit of its partial sums, $S = \lim_{n\to+\infty} S_n$, exists and is a finite real number S.
- Otherwise, we say the series **diverges**.

1.1 Preliminaries and Definitions

A infinite series is an expression of the form

$$\sum_{n=1}^{+\infty} u_n = u_1 + u_2 + u_3 + ... + u_n + ... \tag{1.1}$$

where $(u_n)_{n\geq 1}$ a sequence of real numbers. The number u_n is called the n-th term of the series. An example of a series is

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} + ...$$

The n-th term of this series is $u_n = \frac{1}{2^n}$. To better understand such a series, we introduce the partial sums of the series (1.1). The n-th partial sum S_n of the series is the sum of its n first terms

$$S_n = u_1 + u_2 + u_3 + ... + u_n. \tag{1.2}$$

Each serie is associated with a sequence of partial sums:

$$S_1, S_2, S_3, ..., S_n, ...$$

The sum of the series is defined as the limit of its sequence of partial sums, provided this limit exists.

Remark 1.1.1 *The sum of a series is not a sum in the ordinary sense. It is a limit.*

1.1.1 Nature of a Infinite Series

Definition 1.1.2 *We say that the series* $\sum_{n=1}^{\infty} u_n$ *converges (or is convergent) of sum* $S \in \mathbb{R}$ *provided that the limit of its sequence of partial sums*

$$S = \lim_{n \to +\infty} S_n \tag{1.3}$$

exists (and is finite). Otherwise, the series is said to diverge (or be divergent). If a series diverges, it has no sum.

So an infinite sum is a limit of finite sums.

$$S = \sum_{n=1}^{+\infty} u_n = \lim_{N \to +\infty} \sum_{n=1}^{N} u_n$$

provided that this limit exists.

Example 1.1.3 *Show that the series*

$$\sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$$

converge and find its sum.

Solution. The first four sums are

$$S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}, S_4 = \frac{15}{16}.$$

It seems likely that $S_n = \dfrac{2^n - 1}{2^n}$, and indeed its follows easily by recurrence on n, since

$$\begin{aligned} S_{n+1} = S_n + \frac{1}{2^{n+1}} &= \frac{2^n - 1}{2^n} + \frac{1}{2^{n+1}} \\ &= \frac{2^{n+1} - 2 + 1}{2^{n+1}} = \frac{2^{n+1} - 1}{2^{n+1}}. \end{aligned}$$

So the sum of the given series is

$$S = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{2^n - 1}{2^n} = \lim_{n \to +\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

Example 1.1.4 *The series*

$$\sum_{n=0}^{\infty} (-1)^n \quad \textit{and} \quad \sum_{n=0}^{\infty} 2^n$$

illustrate two forms of divergence: bounded divergence, unbounded divergence.

Solution. For the first series,

$$S_n = 1 - 1 + 1 - 1 + \cdots + (-1)^n$$

Here

$$S_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

The sequence of partial sums reduces to $1, 0, 1, 0, \cdots$. Since the sequence diverges, the series diverges. This is an example of bounded divergence. For the second series,

$$S_n = \sum_{k=0}^{n} 2^k = 1 + 2 + 2^2 + \cdots + 2^n$$

Since $S_n > 2^n$, the sum tends to ∞, and the series diverges. This is an example of unbounded divergence.

Example 1.1.5 *Show that the series*

$$\sum_{n=1}^{+\infty} \frac{1}{4n^2 - 1} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + ...$$

converge and determine its sum.

Solution. We need a formula for the nth partial sum S_n so that we can evaluate the limit when $n \to +\infty$. To find such a formula, we start by observing that

$$u_n = \frac{1}{4n^2 - 1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

It follows that

$$\begin{aligned} S_n &= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots \\ &+ \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1} \end{aligned}$$

Therefore

$$\sum_{n=1}^{+\infty} \frac{1}{4n^2 - 1} = \lim_{n \to +\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Example 1.1.6 *Study the nature of the following series:*

$$\sum_{n \geq 1} \ln\left(1 + \frac{1}{n}\right) = \ln 2 + \ln\left(1 + \frac{1}{2}\right) + \cdots + \ln\left(1 + \frac{1}{n}\right) + \cdots$$

Solution. The sequence of partial sums is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \ln\left(1 + \frac{1}{k}\right) \\ &= \sum_{k=1}^{n} \ln\left(\frac{1+k}{k}\right) = \sum_{k=1}^{n} [\ln(1+k) - \ln(k)] \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(1+n) - \ln(n)) \\ &= \ln(1+n). \end{aligned}$$

Where,

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \ln(1+n) = +\infty.$$

This proves that the series $\sum_{n \geq 1} \ln\left(1 + \frac{1}{n}\right)$ is divergent.

Remark 1.1.7 *The sum of S_n in example 1.1.5 is called a telescopic sum and provides us with a way of finding the sums of certain series. The series in examples 5.1.9 and 5.2.1 are examples of a type of geometric series and the alternating series.*

1.2 Definitions - Geometric Series

Definition 1.2.1 *The series* $\sum_{n=0}^{+\infty} u_n$ *is said to be a geometric series if each term after the first is a fixed multiple of the term preceding it; that is, if there exists a number* q *called the ratio (the base, the reason) of the series such that*

$$u_{n+1} = qu_n$$

for all $n \geq 0$.

So any geometric series takes the form

$$u_0 + qu_0 + q^2u_0 + q^3u_0 + ... = \sum_{n=0}^{+\infty} q^n u_0. \tag{1.4}$$

Note that it's convenient to start the summation at $n = 0$, and so we consider the sum

$$S_n = u_0(1 + q + q^2 + ... + q^n)$$

of the $n + 1$ first terms as the n-th partial sum of the series

Example 1.2.2 *The series*

$$\sum_{n=0}^{+\infty} \frac{2}{3^n} = 2 + \frac{2}{3} + \frac{2}{9} + ... + \frac{2}{3^n} + ...$$

is a geometric series with first term $u_0 = 2$ *and reason* $q = \frac{1}{3}$.

Theorem 1.2.3 (Sum of a geometric series) *If* $|q| < 1$ *then the geometric series in (1.4) converges, and its sum is*

$$S = \sum_{n=0}^{+\infty} q^n u_0 = \frac{u_0}{1-q} \tag{1.5}$$

If $|q| \geq 1$ *and* $u_0 \neq 0$, *then the geometric series diverges.*

Proof: If $q = 1$, then $S_n = (n+1)u_0$, so the series certainly diverges. If $q = -1$ then it diverges by an argument like the one in example 1.1.5. Let's assume that $q \neq 1$ and $q \neq -1$. Then the elementary identity

$$1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1-q}$$

follows after multiplying each member by $1 - q$. Therefore

$$S_n = (1 + q + q^2 + \cdots + q^n)u_0 = u_0 \left(\frac{1}{1-q} - \frac{q^{n+1}}{1-q} \right).$$

If $|q| < 1$, then $\lim_{n \to +\infty} q^{n+1} = 0$. So in this case

$$S = \lim_{n \to +\infty} u_0 \left(\frac{1}{1-q} - \frac{q^{n+1}}{1-q} \right) = \frac{u_0}{1-q}.$$

If $|q| > 1$, then q^{n+1} does not exist and therefore $\lim_{n \to +\infty} S_n$ does not exist. This completes the proof of the theorem. □

Example 1.2.4

1. *Geometric series of reason* $q = \frac{1}{2}$: $\displaystyle\sum_{k=0}^{+\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2.$

2. *Geometric series of reason* $q = \frac{1}{3}$, *with first term* $\frac{1}{3^3}$. *We return to the geometric series starting at* $k = 0$ *by adding and subtracting the first terms:* $\displaystyle\sum_{k=3}^{+\infty} \frac{1}{3^k} = \sum_{k=0}^{+\infty} \frac{1}{3^k} - 1 - \frac{1}{3} - \frac{1}{3^2} = \frac{1}{1-\frac{1}{3}} - \frac{13}{9} = \frac{3}{2} - \frac{13}{9} = \frac{1}{18}.$

3. *Calculating the sum of a series from* $k = 0$ *is purely conventional. You can always change the index to sum from 0. Another way to calculate the same series* $\displaystyle\sum_{k=3}^{+\infty} \frac{1}{3^k}$ *as above is to make the index change* $n = k - 3$ *(and thus* $k = n + 3$ *):*

$$\sum_{k=3}^{+\infty} \frac{1}{3^k} = \sum_{n=0}^{+\infty} \frac{1}{3^{n+3}} = \sum_{n=0}^{+\infty} \frac{1}{3^3} \frac{1}{3^n} = \frac{1}{3^3} \sum_{n=0}^{+\infty} \frac{1}{3^n} = \frac{1}{27} \frac{1}{1-\frac{1}{3}} = \frac{1}{18}$$

4. $\displaystyle\sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{2}\right)^{2k} = \sum_{k=0}^{+\infty} \left(-\frac{1}{4}\right)^k = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}.$

1.2.1 Convergent Series

The convergence of a series does not depend on its first terms: changing a finite number of terms in a series does not change whether it is convergent or divergent. On the other hand, if it is convergent, its sum is obviously modified. A practical way of studying the convergence of a series is to study its remainder: the remainder of order n of a convergent series $\displaystyle\sum_{k=0}^{+\infty} u_k$ is :

$$R_n = u_{n+1} + u_{n+2} + \cdots = \sum_{k=n+1}^{+\infty} u_k$$

Proposition 1.2.5 *If a series is convergent, then* $S = S_n + R_n$ *(for all* $n \geqslant 0$ *) and* $\displaystyle\lim_{n\to+\infty} R_n = 0$.

Proof: We have $S = \displaystyle\sum_{k=0}^{+\infty} u_k = \sum_{k=0}^{n} u_k + \sum_{k=n+1}^{+\infty} u_k = S_n + R_n$. So $R_n = S - S_n \to S - S = 0$ when $n \to +\infty$. □

1.2.2 Sequences and Series

There's no difference between studying sequences and series. It's easy to switch from one to the other. First of all, let's remember that to a series $\displaystyle\sum_{k\geqslant 0} u_k$, we associate the partial sum $S_n = \displaystyle\sum_{k=0}^{n} u_k$ and that by definition the series is convergent if the sequence $(S_n)_{n\geqslant 0}$ converges. Reciprocally, if we want to study a sequence $(a_k)_{k\geqslant 0}$ we can use the following result:

Proposition 1.2.6 *A telescopic sum is a series of the form*

$$\sum_{k\geqslant 0}(a_{k+1}-a_k)$$

This series is convergent if and only if $\ell := \lim_{k\to+\infty} a_k$ *exists and in this case we have:*

$$\sum_{k=0}^{+\infty}(a_{k+1}-a_k)=\ell-a_0$$

Proof:

$$\begin{aligned}S_n &= \sum_{k=0}^{n}(a_{k+1}-a_k)\\ &= (a_1-a_0)+(a_2-a_1)+(a_3-a_2)+\cdots+(a_{n+1}-a_n)\\ &= -a_0+a_1-a_1+a_2-a_2+\cdots+a_n-a_n+a_{n+1}\\ &= a_{n+1}-a_0\end{aligned}$$

□

This is a very important example for what follows.

Example 1.2.7 *The series*

$$\sum_{k=0}^{+\infty}\frac{1}{(k+1)(k+2)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$

is convergent and has the value 1.
Indeed, it can be written as a telescopic sum, and more precisely the partial sum verifies :

$$S_n=\sum_{k=0}^{n}\frac{1}{(k+1)(k+2)}=\sum_{k=0}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=1-\frac{1}{n+2}\to 1 \qquad \text{when } n\to+\infty$$

By changing the index, we also have that the series $\sum_{k=1}^{+\infty}\frac{1}{k(k+1)}$ *and* $\sum_{k=2}^{+\infty}\frac{1}{k(k-1)}$ *are convergent and have the same sum 1.*

1.2.3 Necessary Condition for Convergence

A necessary condition of convergence, elementary but often useful to justify the divergence of a series, is given by the following result:

Theorem 1.2.8 *If the series* $\sum_{k\geqslant 0} u_k$ *converges, then the sequence of general terms* $(u_k)_{k\geqslant 0}$ *tends to 0.*

Remark 1.2.9 *The key point is that we find the general term from the partial sums by the formula*

$$u_n=S_n-S_{n-1}.$$

Proof: Suppose that the series converges. Then the partial sums tend to some number L :

$$S_n = \sum_{k=0}^{n} u_k \to L$$

Only one step behind, the S_{n-1} also tend to L: $S_{n-1} \to L$. Since $u_n = S_n - S_{n-1}$, we have $u_n \to L - L = 0$. A change in notation gives $u_k \to 0$.

$$\text{if} \quad u_k \nrightarrow 0 \quad \text{then} \quad \sum_{k=0}^{\infty} u_k \quad \text{diverges.}$$

This is a very useful observation. □

For example, the series $\displaystyle\sum_{k\geqslant 1}\left(1+\frac{1}{k}\right)$ and $\displaystyle\sum_{k\geqslant 1} k^2$ are divergent.

Example 1.2.10 *The series*

$$\sum_{n=1}^{+\infty}(-1)^{n-1}n^2 = 1 - 4 + 9 - 16 + 25 - ...$$

diverges because $\lim\limits_{n\to+\infty} u_n$ *does not exist, while the series*

$$\sum_{n=1}^{+\infty} \frac{n}{3n+1} = \frac{1}{4} + \frac{2}{7} + \frac{3}{10} + \frac{4}{13} + ...$$

diverges because $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \dfrac{n}{3n+1} = \dfrac{1}{3} \neq 0.$

Warning. The reciprocal of the theorem 1.2.8 is false. The condition $\lim\limits_{n\to+\infty} u_n = 0$ is necessary but not sufficient for the convergence of the series $\sum u_n$. That is, a series can satisfy the condition $\lim\limits_{n\to+\infty} u_n = 0$ and yet diverge. An important example of a divergent series with terms approaching zero is the harmonic series :

$$\sum_{k=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

Example 1.2.11

1. *As* $k \to \infty$, $\dfrac{k}{k+1} \to 1$. *Since* $\dfrac{k}{k+1} \nrightarrow 0$, *the series*

$$\sum_{k=0}^{\infty} \frac{k}{k+1} = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots \qquad \textit{diverges.}$$

2. *Since* $\sin k \nrightarrow 0$ *as* $k \to \infty$, *the series*

$$\sum_{k=0}^{\infty} \sin k = \sin 0 + \sin 1 + \sin 2 + \sin 3 + \cdots \qquad \textit{diverges}$$

CAUTION Theorem 1.2.8 does not say that, if $u_k \to 0$, then $\displaystyle\sum_{k=0}^{\infty} u_k$ converges. There are divergent series for which $u_k \to 0$. (One such series appears in Example 1.2.12.)

Example 1.2.12 *The k th term of the series*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

tends to zero:

$$u_k = \frac{1}{\sqrt{k}} \to 0 \quad as \quad k \to \infty$$

However, the series diverges:

$$S_n = \frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} \geq \underbrace{\frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \ terms} = \frac{n}{\sqrt{n}} = \sqrt{n} \to \infty$$

Proposition 1.2.13 *The deletion (or addition) of a finite number of terms in a series affects neither the convergence nor the divergence of that series. This is still equivalent to*

$$\sum_{n\geq 0} u_n \ converge \ \Leftrightarrow \sum_{n=p+1}^{+\infty} u_n \ converge \ \forall p \geq 1.$$

Proof: Consider the following two series:

$$\sum_{n\geq 0} u_n = u_0 + u_1 + u_2 + ... + u_n + ... \tag{1.6}$$

$$\sum_{n=p+1}^{+\infty} u_n = u_{p+1} + u_{p+2} + u_{p+3} + ... + u_{p+n} + ..., \forall p \geq 1 \tag{1.7}$$

Let $S_n = \sum_{k=0}^{n} u_k$ be the partial sum of (1.6) and $L_n = \sum_{k=1}^{n} u_{p+k}$ be the partial sum of (1.7). Then we have

$$S_{n+p} = \sum_{k=0}^{n+p} u_k = \sum_{k=0}^{p} u_k + \sum_{k=p+1}^{n+p} u_k = S_p + L_n.$$

Where

$$L_n = S_{n+p} - S_p \ \forall p \text{ fixed}. \tag{1.8}$$

Therefore, if $\sum_{n\geq 0} u_n$ converges then $\lim_{n\to+\infty} S_n = S$, which also implies that

$$\lim_{n\to+\infty} S_{n+p} = S \tag{1.9}$$

From (1.8) and (1.9), we get

$$\lim_{n\to+\infty} L_n = S - S_p.$$

Where $\sum_{n=p+1}^{+\infty} u_n$ is convergent. Conversely, if the series $\sum_{n=p+1}^{+\infty} u_n$ converges then

$$\lim_{n\to+\infty} L_n = L. \tag{1.10}$$

Applying (1.8) to (1.10), we obtain

$$\lim_{n\to+\infty} S_{n+p} = L + S_p.$$

Therefore,

$$\lim_{n\to+\infty} S_n = L + S_p.$$

From where $\sum_{n\geq 0} u_n$ is convergent. □

1.2.4 Some Basic Results

Theorem 1.2.14

1. *If* $\sum_{k=0}^{\infty} a_k$ *converges and* $\sum_{k=0}^{\infty} b_k$ *converges, then* $\sum_{k=0}^{\infty}(a_k+b_k)$ *converges. Moreover, if* $\sum_{k=0}^{\infty} a_k = L$ *and* $\sum_{k=0}^{\infty} b_k = M$*, then* $\sum_{k=0}^{\infty}(a_k+b_k) = L+M$*.*

2. *If* $\sum_{k=0}^{\infty} a_k$ *converges, then* $\sum_{k=0}^{\infty} \alpha a_k$ *converges for each real number* α*. Moreover, if* $\sum_{k=0}^{\infty} a_k = L$*, then* $\sum_{k=0}^{\infty} \alpha a_k = \alpha L$*.*

Proof: Let

$$S_n = \sum_{k=0}^{n} a_k, \quad T_n = \sum_{k=0}^{n} b_k, \quad U_n = \sum_{k=0}^{n}(a_k+b_k), \quad V_n = \sum_{k=0}^{n} \alpha a_k$$

Note that

$$U_n = S_n + T_n \quad \text{and} \quad V_n = \alpha S_n$$

If $S_n \to L$ and $T_n \to M$, then

$$U_n \to L+M \quad \text{and} \quad V_n \to \alpha L.$$

□

For example:

$$\sum_{k=0}^{+\infty}\left(\frac{1}{2^k}+\frac{5}{3^k}\right) = \sum_{k=0}^{+\infty}\frac{1}{2^k} + 5\sum_{k=0}^{+\infty}\frac{1}{3^k} = \frac{1}{1-\frac{1}{2}} + 5\frac{1}{1-\frac{1}{3}} = 2+5\frac{3}{2} = \frac{19}{2}.$$

As an application to series with complex terms, convergence is equivalent to that of the real and imaginary parts:

Proposition 1.2.15 *Let* $(u_k)_{k\geqslant 0}$ *be a sequence of complex numbers. For all* k*, let* $u_k = a_k + \mathrm{i} b_k$*, with* a_k *the real part of* u_k *and* b_k *the imaginary part. The series* $\sum u_k$ *converges if and only if both series* $\sum a_k$ *and* $\sum b_k$ *converge. If this is the case, we have:*

$$\sum_{k=0}^{+\infty} u_k = \sum_{k=0}^{+\infty} a_k + \mathrm{i}\sum_{k=0}^{+\infty} b_k.$$

Proof: Let S_n, A_n and B_n be the partial sums defined by:

$$S_n = \sum_{k=0}^{n} u_k, \quad A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k.$$

For any $n \in \mathbb{N}$, we have:

$$S_n = \sum_{k=0}^{n} (a_k + ib_k) = \sum_{k=0}^{n} a_k + i \sum_{k=0}^{n} b_k = A_n + iB_n.$$

- **Necessity:** Assume the series $\sum u_k$ converges. Then the sequence (S_n) converges to some complex number $S = \alpha + i\beta$. Since

  $$A_n = \text{Re}(S_n) \quad \text{and} \quad B_n = \text{Im}(S_n),$$

  and the real and imaginary parts are continuous functions, we have:

  $$\lim_{n\to\infty} A_n = \text{Re}(S) = \alpha \quad \text{and} \quad \lim_{n\to\infty} B_n = \text{Im}(S) = \beta.$$

  Hence both series $\sum a_k$ and $\sum b_k$ converge.

- **Sufficiency:** Conversely, assume that both series $\sum a_k$ and $\sum b_k$ converge. Let:

  $$A = \sum_{k=0}^{+\infty} a_k \quad \text{and} \quad B = \sum_{k=0}^{+\infty} b_k.$$

  Then $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$. Using the algebraic properties of limits in $\mathbb{C}$, we obtain:

  $$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (A_n + iB_n) = A + iB.$$

  Therefore, the series $\sum u_k$ converges.

In either case, when the series converge, we have:

$$\sum_{k=0}^{+\infty} u_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} A_n + i \lim_{n\to\infty} B_n = \sum_{k=0}^{+\infty} a_k + i \sum_{k=0}^{+\infty} b_k,$$

which completes the proof. □

Example 1.2.16 *Consider the complex geometric series* $\sum_{k=0}^{+\infty} r^k$*, where* $r = \rho e^{i\theta}$ *is a complex number with modulus* $\rho < 1$ *and argument* $\theta \in \mathbb{R}$.

Since $|r| = \rho < 1$*, the geometric series converges and its sum is:*

$$\sum_{k=0}^{+\infty} r^k = \frac{1}{1-r}.$$

Using de Moivre's formula, we can write:

$$r^k = \rho^k e^{ik\theta} = \rho^k \big(\cos(k\theta) + i\sin(k\theta)\big).$$

Hence, the real and imaginary parts of r^k *are respectively:*

$$a_k = \rho^k \cos(k\theta) \quad \textit{and} \quad b_k = \rho^k \sin(k\theta).$$

By the previous proposition, since the series $\sum_{k=0}^{+\infty} r^k$ *converges, the two real series* $\sum_{k=0}^{+\infty} a_k$ *and* $\sum_{k=0}^{+\infty} b_k$ *also converge, and we have:*

$$\sum_{k=0}^{+\infty} \rho^k \cos(k\theta) = \mathrm{Re}\left(\frac{1}{1-r}\right) \quad \text{and} \quad \sum_{k=0}^{+\infty} \rho^k \sin(k\theta) = \mathrm{Im}\left(\frac{1}{1-r}\right).$$

To obtain explicit formulas, we compute the real and imaginary parts of $\frac{1}{1-r}$*:*

$$\frac{1}{1-r} = \frac{1}{1-\rho e^{i\theta}} = \frac{1}{1-\rho\cos\theta - i\rho\sin\theta} = \frac{1-\rho\cos\theta + i\rho\sin\theta}{(1-\rho\cos\theta)^2 + (\rho\sin\theta)^2}.$$

Simplifying the denominator:

$$(1-\rho\cos\theta)^2 + (\rho\sin\theta)^2 = 1 - 2\rho\cos\theta + \rho^2\cos^2\theta + \rho^2\sin^2\theta = 1 + \rho^2 - 2\rho\cos\theta.$$

Therefore,

$$\frac{1}{1-r} = \frac{1-\rho\cos\theta}{1+\rho^2-2\rho\cos\theta} + i\,\frac{\rho\sin\theta}{1+\rho^2-2\rho\cos\theta}.$$

Finally, we obtain the closed-form expressions:

$$\boxed{\sum_{k=0}^{+\infty} \rho^k \cos(k\theta) = \frac{1-\rho\cos\theta}{1+\rho^2-2\rho\cos\theta}}, \qquad \boxed{\sum_{k=0}^{+\infty} \rho^k \sin(k\theta) = \frac{\rho\sin\theta}{1+\rho^2-2\rho\cos\theta}}.$$

These formulas hold for any $\rho \in [0,1)$ *and* $\theta \in \mathbb{R}$.

1.2.5 Sums of Series

For the moment, there aren't many series for which you know the sum, apart from geometric series. We'll have to wait for other chapters and other techniques to calculate series sums. In this chapter, we'll focus on whether a series converges or diverges. But here's an exception!

Example 1.2.17 *Let* $q \in \mathbb{C}$ *be such that* $|q| < 1$*. What is the sum*

$$\sum_{k=0}^{+\infty} kq^k$$

Let's assume for a moment that this series converges and denote $S = \sum_{k=0}^{+\infty} kq^k$*. Let's write:*

$$\begin{aligned}
S = \sum_{k=0}^{+\infty} kq^k &= \sum_{k=1}^{+\infty} kq^k = q\sum_{k=1}^{+\infty} kq^{k-1} \\
&= q\sum_{k=1}^{+\infty} q^{k-1} + q\sum_{k=1}^{+\infty} (k-1)q^{k-1} \\
&= q\sum_{k=1}^{+\infty} q^{k-1} + q\sum_{k'=0}^{+\infty} k'q^{k'} \qquad \textit{posing } k' = k-1 \\
&= q\sum_{k=1}^{+\infty} q^{k-1} + q\cdot S
\end{aligned}$$

Solving this equation in S, we find that

$$(1-q)S = q\sum_{k=1}^{+\infty} q^{k-1}$$

This last series is a geometric series of reason q with $|q|<1$, so it converges. This justifies the convergence of S. Conclusion:

$$(1-q)S = q\cdot\frac{1}{1-q}$$

$$S=\sum_{k=0}^{+\infty} kq^k = \frac{q}{(1-q)^2}.$$

1.2.6 Cauchy Convergence Test for Series

Caution! There are series $\sum_{k\geqslant 0} u_k$ such that $\lim_{k\to+\infty} u_k = 0$, but $\sum_{k\geqslant 0} u_k$ diverges. The most classic example is harmonic series:

$$\text{The series } \sum_{k\geqslant 1}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \quad \text{diverges}$$

More precisely, we have $\lim_{n\to+\infty} S_n = +\infty$. However, we have $u_k = \dfrac{1}{k}\to 0$ (when $k\to+\infty$). To show that the series diverges, we'll use the Cauchy criterion. Recall. A (u_n) sequence of real numbers converges if and only if it is a Cauchy sequence, i.e. :

$$\forall\varepsilon>0 \quad \exists n_0\in\mathbb{N} \quad \forall m,n\geqslant n_0 \quad |u_n-u_m|<\varepsilon$$

For series, this gives us :

Theorem 1.2.18 (Cauchy convergence test for series.) *The sum $\sum_{k=0}^{+\infty} u_k$ converges if and only if verifies the following Cauchy criterion:*

$$\forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall m,n\geqslant n_0, \left|\sum_{k=n+1}^{m} u_k\right|<\varepsilon$$

Proof: We have $\sum_{n\geqslant 0} u_n$ is convergent $\Leftrightarrow (S_n)$ is convergent $\Leftrightarrow (S_n)$ is Cauchy since $\mathbb{R}$ is complete. Therefore,

$$\forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall n,m\in\mathbb{N} : (n,m>n_0 \Rightarrow |S_n-S_m|<\varepsilon),$$

where $S_n=\sum_{k=0}^{n} u_k$. And as

$$|S_n-S_m| = \left|\sum_{k=0}^{n} u_k - \sum_{k=0}^{m} u_k\right| = \left|\sum_{k=n+1}^{m} u_k\right|.$$

Therefore,

$$\forall\varepsilon>0, \exists n_0\in\mathbb{N}, \forall n,m\in\mathbb{N} : (n,m>n_0 \Rightarrow \left|\sum_{k=n+1}^{m} u_k\right|<\varepsilon).$$

Where the result. ☐

Let's return to the harmonic series $\sum_{k\geqslant 1}\frac{1}{k}$. The partial sum is $S_n=\sum_{k=1}^{n}\frac{1}{k}$. Let's calculate the difference of two partial sums, in order to keep the terms between $n+1$ (which plays the role of n) and $2n$ (which plays the role of m):

$$S_{2n}-S_n=\frac{1}{n+1}+\cdots+\frac{1}{2n}\geqslant\frac{n}{2n}=\frac{1}{2}$$

The sequence of partial sums is not Cauchy (because $\frac{1}{2}$ is not less than $\varepsilon=\frac{1}{4}$ for example), so the series does not converge. If you wish to complete the demonstration without directly using Cauchy's criterion, then reason by the absurd. Suppose $S_n\to\ell\in\mathbb{R}$ (when $n\to+\infty$). Then we also have $S_{2n}\to\ell$ (when $n\to+\infty$) and therefore $S_{2n}-S_n\to\ell-\ell=0$. This contradicts the inequality $S_{2n}-S_n\geqslant\frac{1}{2}$.

We end with a further study of the harmonic series.

Proposition 1.2.19 *For the harmonic series* $\sum_{k\geqslant 1}\frac{1}{k}$ *and its partial sum* $S_n=\sum_{k=1}^{n}\frac{1}{k}$*, we have*

$$\lim_{n\to+\infty}S_n=+\infty$$

Proof: Let $M>0$. Choose $m\in\mathbb{N}^*$ such that $m\geqslant 2M$. Then for $n\geqslant 2^m$ we have :

$$\begin{aligned}
S_n&=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m}+\cdots+\frac{1}{n}\\
&\geqslant 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m}\\
&=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots+\left(\frac{1}{2^{m-1}+1}+\cdots+\frac{1}{2^m}\right)\\
&\geqslant 1+\frac{1}{2}+2\frac{1}{4}+4\frac{1}{8}+8\frac{1}{16}+\cdots+2^{m-1}\frac{1}{2^m}\\
&=1+m\frac{1}{2}\geqslant M
\end{aligned}$$

The trick is to group the terms together. Between each parenthesis there are successively $2,4,8,\ldots$ terms up to

$$2^m-\left(2^{m-1}+1\right)+1=2^m-2^{m-1}=2^{m-1}\quad\text{terms.}$$

Thus for any $M>0$ there exists $n_0\geqslant 0$ such that, for any $n\geqslant n_0$, we have $S_n\geqslant M$; thus (S_n) tends to $+\infty$. This of course means that the harmonic series diverges. ☐

Example 1.2.20 *Using Cauchy's criterion, we show that the* $\sum_{n\geq 0}\frac{\sin n}{2^n}$ *series is convergent. In fact, we'll show the following Cauchy property:*

$$\forall\varepsilon>0\ \exists n_0\in\mathbb{N},\ \forall m,n\geqslant n_0,\ \left|\sum_{k=n+1}^{m}\frac{\sin k}{2^k}\right|<\varepsilon$$

Let $\varepsilon>0$. *For* $n,m\in\mathbb{N}$, *we have*

$$\left|\sum_{k=n+1}^{m}\frac{\sin k}{2^k}\right|\leq\sum_{k=n+1}^{m}\left|\frac{\sin k}{2^k}\right|\leq\sum_{k=n+1}^{m}\frac{1}{2^k}.\tag{1.11}$$

Or,

$$\sum_{k=n+1}^{m} \frac{1}{2^k} = \frac{1}{2^{n+1}} \frac{1-\left(\frac{1}{2}\right)^{m-n}}{1-\frac{1}{2}} = \frac{1}{2^n}\left(1-\frac{1}{2^{m-n}}\right). \tag{1.12}$$

Furthermore, if $m > n$

$$1-\left(\frac{1}{2}\right)^{m-n} < 1.$$

And therefore,

$$\frac{1}{2^n}\left(1-\left(\frac{1}{2}\right)^{m-n}\right) < \frac{1}{2^n}. \tag{1.13}$$

From (1.11)-(1.13), we get

$$\left|\sum_{k=n+1}^{m} \frac{\sin k}{2^k}\right| \leq \frac{1}{2^n}.$$

Thus, for $\left|\sum_{k=n+1}^{m} \frac{\sin k}{2^k}\right|$ *to be smaller than ε, it suffices that*

$$\frac{1}{2^n} < \varepsilon,$$

which is equivalent to

$$n > \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}$$

Thus, it suffices to take $n_0 = \left\lfloor \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2} \right\rfloor + 1.$

1.3 Infinite Series with Positive Terms

Infinite series with positive or zero terms behave like increasing sequences and are therefore easier to study.

1.3.1 Convergence by Partial Sums

Recall. Let $(u_n)_{n\geqslant 0}$ be an increasing sequence of real numbers.

- If the sequence is bounded, then the sequence (u_n) converges, i.e. it admits a finite limit.
- Otherwise the sequence (u_n) tends to $+\infty$.

Let's apply this to the $\sum u_k$ series with positive terms, i.e. $u_k \geqslant 0$ for all k. In this case, the sequence (S_n) of partial sums, defined by $S_n = \sum_{k=0}^{n} u_k$, is an increasing sequence. This is because

$$S_n - S_{n-1} = u_n \geqslant 0.$$

From the reminders on sequences, we have :

Proposition 1.3.1 *A series with positive terms is a convergent series if and only if the sequence of partial sums is bounded above. In other words, if and only if there exists $M > 0$ such that, for any $n \geqslant 0, S_n \leqslant M$.*

Proof: We show the implication ($\Leftarrow$). Assume that the sequence of partial sums (S_n) of $\sum_{n\geq 0} u_n$ is bounded above, and show that $\sum_{n\geq 0} u_n$ is convergent. We have

$$S_n = u_0 + u_1 + ... + u_n = S_{n-1} + u_n.$$

Therefore

$$u_n = S_n - S_{n-1}.$$

Thus,

$$S_n \geq S_{n-1} \quad \forall n$$

since $u_n \geq 0$. Where, (S_n) is increasing. So (S_n) is increasing and bounded above, so it converges. This shows that the series $\sum_{n\geq 0} u_n$ is convergent. Conversely, by hypothesis $\sum_{n\geq 0} u_n$ is convergent, so the series of partial sums (S_n) is convergent. Therefore (S_n) is bounded, where bounded above. □

Remark 1.3.2 *In the case of convergence, the sum of the S series of course verifies* $\lim S_n = S$, *but also $S_n \leqslant S$, for all n. Both convergence/divergence situations are possible: $\sum_{k\geqslant 0} q^k$ converges if $0 < q < 1$, and diverges if $q \geqslant 1$.*

Example 1.3.3 *Consider the following positive-term series:*

$$\sum_{n\geq 0} u_n, \text{ where } u_n = \frac{1}{n^2} \; \forall n \in \mathbb{N}^*.$$

We know that

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} \; \forall n > 1$$

Where

$$\begin{aligned} S_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}, \\ &\leq 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right), \\ &\leq 2 - \frac{1}{n} < 2. \end{aligned}$$

Thus $(S_n)_n$ is bounded above by 2. Therefore $\sum_{n\geq 1} \frac{1}{n^2}$ is convergent.

Corollary 1.3.4 *The series $\sum_{n\geq 1} u_n$, with $u_n \geq 0$ diverges if and only if $\lim_{n\to+\infty} S_n = +\infty$.*

Proof: The proof follows from the fact that $(S_n)_n$ is increasing but not bounded above. □

1.4 Basic Comparison, Limit Comparison; The Integral Test

1.4.1 The Basic Comparison Theorem

What is the general method for finding the nature of a series with positive terms? We compare it with simple classical series using the following comparison theorem.

Theorem 1.4.1 (Theorem of comparison) *Let* $\sum u_k$ *and* $\sum v_k$ *be two series with positive or zero terms. It is assumed that there exists* $k_0 \geqslant 0$ *such that, for any* $k \geqslant k_0$, $u_k \leqslant v_k$.

- *If* $\sum v_k$ *converges then* $\sum u_k$ *converges.*
- If $\sum u_k$ *diverges then* $\sum v_k$ *diverges.*

Proof: As we have seen, convergence does not depend on the first terms. Without loss of generality, we can therefore assume $k_0 = 0$. Let's note $S_n = u_0 + \cdots + u_n$ and $S'_n = v_0 + \cdots + v_n$. The sequences (S_n) and $\left(S'_n\right)$ are increasing, and furthermore, for any $n \geqslant 0, S_n \leqslant S'_n$. If the series $\sum v_k$ converges, then the series $\left(S'_n\right)$ converges. Let S' be its limit. The series (S_n) is increasing and bounded above by S', so it converges, and so the series $\sum u_k$ also converges. Conversely, if the series $\sum u_k$ diverges, then the series (S_n) tends to $+\infty$, and the same applies to the series $\left(S'_n\right)$ and so the series $\sum v_k$ diverges. □

We now present the second comparison theorem.

Theorem 1.4.2 *Let* $\sum_{n\geq 0} u_n$, $\sum_{n\geq 0} v_n$ *be two series with positive terms. If there exist* $a > 0$, $b > 0$ *and a natural number* n_0 *such that*

$$\forall n \geq n_0,\ a \leq \frac{u_n}{v_n} \leq b,$$

with $v_n > 0$. *Then the two series are of the same nature.*

Proof: The proof follows from the fact that we have

$$av_n \leq u_n \leq bv_n\ \forall n \in \mathbb{N}.$$

Indeed, if $\sum_{n\geq 0} u_n$ converges then according to the theorem 1.4.1, $\sum_{n\geq 0} av_n$ converges. Therefore $\sum_{n\geq 0} v_n$ converges. And if $\sum_{n\geq 0} v_n$ converges then $\sum_{n\geq 0} bv_n$ converges. Using Theorem 1.4.1, $\sum_{n\geq 0} u_n$ converges. Thus,

$$\sum_{n\geq 0} u_n \text{ converge} \Leftrightarrow \sum_{n\geq 0} v_n \text{ converge.}$$

Where the two series are of the same nature. □

We then have the third comparison theorem.

Theorem 1.4.3 *Let* $\sum_{n\geq 0} u_n$, $\sum_{n\geq 0} v_n$ *be two series with positive terms such that* $\lim_{n\to+\infty} \frac{u_n}{v_n} = l$.

1. *If* $0 < l < +\infty$ *then the two series are of the same nature, and in this case the series are said to be* l*-equivalent.*
2. *If* $l = 0$ *and if* $\sum_{n\geq 0} v_n$ *converges then* $\sum_{n\geq 0} u_n$ *converges.*
3. *If* $l = +\infty$ *and if* $\sum_{n\geq 0} u_n$ *converges then* $\sum_{n\geq 0} v_n$ *converges.*

Proof:

1. We assume that $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = l$ then by definition of the limit, we get

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : \left(n \geq n_0 \Rightarrow \left|\frac{u_n}{v_n} - l\right| < \varepsilon\right).$$

   This means that for any $n \geq n_0$, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon.$$

   Therefore,

$$a \leq \frac{u_n}{v_n} \leq b,$$

   where $a = l - \varepsilon$ and $b = l + \varepsilon$. By choosing $0 < \varepsilon < l$, we obtain $a > 0$ and $b > 0$. Thus, by applying the second comparison theorem, we deduce that the two series are of the same nature.

2. A reasoning analogous to the one above gives the result.

3. Follows from the definition of the limit in the case where $l = +\infty$ and the application of the first comparison theorem.

□

1.4.2 Examples

Example 1.4.4 *We've already seen in example 1.2.7 that the series*

$$\sum_{k=0}^{+\infty} \frac{1}{(k+1)(k+2)} \quad \textit{converges.}$$

We deduce that

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} \quad \textit{converges.}$$

In fact, we have :

$$\lim_{k\to+\infty} \frac{\frac{1}{2k^2}}{\frac{1}{(k+1)(k+2)}} = \frac{1}{2}.$$

In particular, there exists k_0 such that for $k \geqslant k_0$:

$$\frac{1}{2k^2} \leqslant \frac{1}{(k+1)(k+2)}$$

In fact, this is true for $k \geqslant 4$, but there's no need to calculate a precise value for k_0. We deduce that the series with general term $\dfrac{1}{2k^2}$ converges, hence the result by linearity.

Example 1.4.5 *Here's a fundamental example, the exponential series.*

$$\textit{The series } \sum_{k\geqslant 0} \frac{1}{k!} \quad \textit{converges.}$$

Note that $0! = 1$ *and that for* $k \geqslant 1, k! = 1 \cdot 2 \cdot 3 \cdots k$.
Indeed $\frac{1}{k!} \leqslant \frac{1}{k(k-1)}$ *for* $k \geq 2$*, but* $\sum_{k\geqslant 2} \frac{1}{k(k-1)} = \sum_{k\geqslant 0} \frac{1}{(k+1)(k+2)}$ *(by index change) is a convergent series. So the exponential series* $\sum_{k\geqslant 0} \frac{1}{k!}$ *converges. In fact, by definition, the sum* $\sum_{k=0}^{+\infty} \frac{1}{k!}$ *is equal to the Euler number* $e = \exp(1)$.

Example 1.4.6 *Conversely, we have seen that the series* $\sum_{k\geqslant 1} \frac{1}{k}$ *diverges. We can easily deduce that the series* $\sum_{k\geqslant 1} \frac{\ln(k)}{k}$ *and* $\sum_{k\geqslant 1} \frac{1}{\sqrt{k}}$ *also diverge.*

Let's finish with an interesting application: the decimal expansion of a real number.

Example 1.4.7 *Let* $(a_k)_{k\geqslant 1}$ *be a sequence of integers all between 0 and 9. The series*

$$\sum_{k=1}^{+\infty} \frac{a_k}{10^k} \text{ converges.}$$

Indeed, its general term $u_k = \frac{a_k}{10^k}$ *is increased by* $\frac{9}{10^k}$. *But the geometric series* $\sum \frac{1}{10^k}$ *converges, because* $\frac{1}{10} < 1$. *The series* $\sum \frac{9}{10^k}$ *also converges by linearity, hence the result.*

Such a sum $\sum_{k=1}^{+\infty} \frac{a_k}{10^k}$ *is a decimal of a real* x*, with* $0 \leqslant x \leqslant 1$.
For example, if $a_k = 3$ *for all* k *:*

$$\sum_{k=1}^{+\infty} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = 0,3 + 0,03 + 0,003 + \cdots = 0,333\ldots = \frac{1}{3}$$

Of course, the same result can be obtained using the geometric series:

$$\sum_{k=1}^{+\infty} \frac{3}{10^k} = \frac{3}{10} \sum_{k=0}^{+\infty} \frac{1}{10^k} = \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{3}{10} \cdot \frac{10}{9} = \frac{1}{3}$$

1.4.3 Theorem of Equivalents

Let's improve the comparison theorem with the notion of equivalent sequences. Let (u_k) and (v_k) be two strictly positive sequences. Then the sequences (u_k) and (v_k) are equivalent if

$$\lim_{k\to+\infty} \frac{u_k}{v_k} = 1$$

Then

$$u_k \sim v_k$$

Theorem 1.4.8 (Theorem of equivalents) *Let* (u_k) *and* (v_k) *be two sequences with strictly positive terms. If* $u_k \sim v_k$ *then the sequences* $\sum u_k$ *and* $\sum v_k$ *are of the same nature.*

In other words, if the sequences are equivalent, then they are either both convergent or both divergent. Of course, if they both converge, there's no reason why the sums should be equal. Finally, if the sequences are both strictly negative, the conclusion remains valid.

Let's return to an example that shows how practical this theorem is: the sequences $\frac{1}{k^2}$ and $\frac{1}{(k+1)(k+2)} = \frac{1}{k^2+3k+2}$ are equivalent. Since the series $\sum \frac{1}{(k+1)(k+2)}$ converges (example 1.2.7), then this implies that $\sum \frac{1}{k^2}$ converges.

Proof: By hypothesis, for any $\varepsilon > 0$, there exists k_0 such that, for any $k \geqslant k_0$,

$$\left|\frac{u_k}{v_k} - 1\right| < \varepsilon,$$

where in other words

$$(1-\varepsilon)v_k < u_k < (1+\varepsilon)v_k.$$

Let's set a $\varepsilon < 1$. If $\sum u_k$ converges, then by the theorem 1.4.1 , $\sum(1-\varepsilon)v_k$ converges, so $\sum v_k$ also converges. Conversely, if $\sum u_k$ diverges, then $\sum(1+\varepsilon)v_k$ diverges, and so does $\sum v_k$. □

1.4.4 Examples

Example 1.4.9 *The two series*

$$\sum \frac{k^2+3k+1}{k^4+2k^3+4} \quad \textit{and} \quad \sum \frac{k+\ln(k)}{k^3} \textit{ convergent.}$$

In both cases, the general term is equivalent to $\frac{1}{k^2}$*, and we know that the series* $\sum \frac{1}{k^2}$ *converges.*

Example 1.4.10 *On the other hand*

$$\sum \frac{k^2+3k+1}{k^3+2k^2+4} \quad \textit{and} \quad \sum \frac{k+\ln(k)}{k^2} \textit{ divergent.}$$

In both cases, the general term is equivalent to $\frac{1}{k}$*, and we've seen that the* $\sum \frac{1}{k}$ *series diverges.*

Example 1.4.11 *Does the series*

$$\sum_{k\geqslant 1} \ln(\ th\ k) \textit{ converge?}$$

The method is to look for a simple equivalent of the general term.

- *Let us first note that, for* $k > 0, 0 < th\ k < 1$.
- *Then let's evaluate th* k *:*

$$th\ k = \frac{\operatorname{sh} k}{\operatorname{ch} k} = \frac{e^k - e^{-k}}{e^k + e^{-k}} = 1 + \frac{-2e^{-k}}{e^k+e^{-k}} = 1 + \frac{-2e^{-2k}}{1+e^{-2k}}$$

- *Since* $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$*, then, if* $u_k \to 0, \ln(1+u_k) \sim u_k$*. Thus*

$$\ln(\ th\ k) = \ln\left(1 + \frac{-2e^{-2k}}{1+e^{-2k}}\right) \sim \frac{-2e^{-2k}}{1+e^{-2k}} \sim -2e^{-2k}$$

- *The series* $\sum e^{-2k} = \sum \left(e^{-2}\right)^k$ *converges because it is a geometric series of reason* $\dfrac{1}{e^2} < 1$.
- *The sequences* $\ln(\ th\ k)$ *and* $-2e^{-2k}$ *are two strictly negative sequences and we have seen that* $\ln(\ th\ k) \sim -2e^{-2k}$*. By the theorem 1.4.8 of equivalents, since the series* $\sum -2e^{-2k}$ *converges, then the series* $\sum \ln(\ th\ k\ )$ *also converges. (If you prefer, you can apply the theorem to the strictly positive sequences* $-\ln(\ th\ k)$ *and* $2e^{-2k}$*.)*

1.4.5 The Integral Test

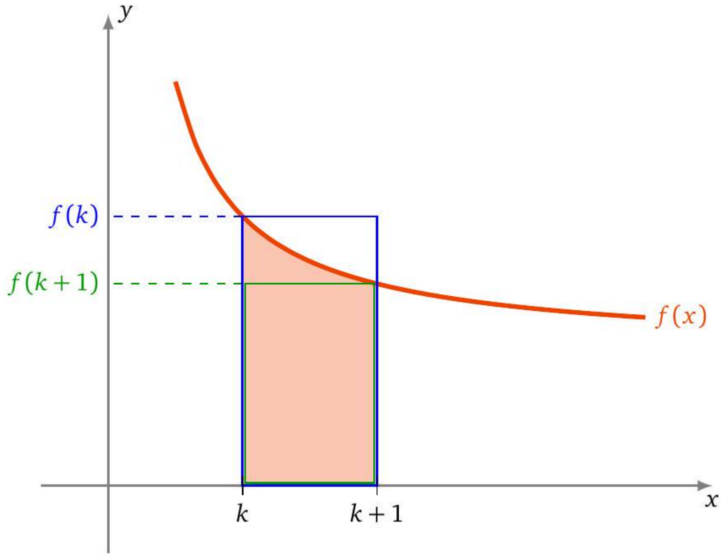
Theorem 1.4.12 *If* f *is continuous, positive, and decreasing on* $[0, +\infty[$*, then*

$$\sum_{k=0}^{\infty} f(k) \quad \textit{converges} \quad \textit{iff} \quad \int_0^{\infty} f(x)dx \quad \textit{converges.}$$

"Of the same nature" means that the series and integral of the theorem are either convergent at the same time, or divergent at the same time. Caution! It's important that f is positive and decreasing. The easiest way is to understand the drawing and repeat the demonstration whenever you need to.

Proof: Let $k \in \mathbb{N}$. Since f is decreasing, for $k \leqslant t \leqslant k+1$, we have $f(k+1) \leqslant f(t) \leqslant f(k)$ (note the order). Integrating over the interval $[k, k+1]$ of length 1, we obtain :

$$f(k+1) \leqslant \int_k^{k+1} f(t)\,\mathrm{d}t \leqslant f(k)$$



In the drawing, this inequality means that the area under the curve, between abscissas k and $k+1$, lies between the area of the green rectangle with height $f(k+1)$ and base 1 and the area of the blue rectangle with height $f(k)$ and the same base 1. We sum these inequalities for k varying from 0 to $n-1$:

$$\sum_{k=0}^{n-1} f(k+1) \leqslant \sum_{k=0}^{n-1} \int_k^{k+1} f(t)\mathrm{d}t \leqslant \sum_{k=0}^{n-1} f(k).$$

Let :

$$u_1 + \cdots + u_n \leqslant \int_0^n f(t)\mathrm{d}t \leqslant u_0 + \cdots + u_{n-1}.$$

The series $\sum u_k$ converges and has sum S if and only if the sequence of partial sums converges to S. If this is the case, $\int_0^n f(t)\mathrm{d}t$ is increased by S, and since $\int_0^x f(t)\mathrm{d}t$ is an increasing function of x (by positivity of f), the integral converges. Conversely, if the integral converges, then $\int_0^n f(t)\mathrm{d}t$ is major, so is the sequence of partial sums, and the series converges. ☐

1.4.6 Applying the Integral Test

Example 1.4.13 (Harmonic series) *The harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad \textit{diverges.}$$

The function $f(x) = 1/x$ is continuous, positive, and decreasing on $[1, \infty)$. We know that

$$\int_1^{\infty} \frac{dx}{x} \qquad \textit{diverges.}$$

By the integral test,

$$\sum_{k=1}^{\infty} \frac{1}{k} \qquad \textit{diverges.}$$

The next example generalizes on this.

Example 1.4.14 (p-series) *The p-series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \qquad \textit{converges} \qquad \textit{iff} \qquad p > 1$$

If $p \le 0$, then the terms of the series are all greater than or equal to 1. Therefore, the terms do not tend to zero and the series cannot converge. We assume therefore that $p > 0$. The function $f(x) = 1/x^p$ is then continuous, positive, and decreasing on $[1, \infty)$. Thus, by the integral test,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \textit{converges} \qquad \textit{iff} \qquad \int_1^{\infty} \frac{dx}{x^p} \qquad \textit{converges.}$$

Earlier you saw that

$$\int_1^{\infty} \frac{dx}{x^p} \qquad \textit{converges} \qquad \textit{iff} \qquad p > 1$$

It follows that

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \textit{converges} \qquad \textit{iff} \qquad p > 1$$

Example 1.4.15 *Show that the series*

$$\sum_{k=1}^{\infty} \frac{1}{k\ln(k+1)} = \frac{1}{\ln 2} + \frac{1}{2\ln 3} + \frac{1}{3\ln 4} + \cdots \qquad \textit{diverges}$$

We begin by setting $f(x) = \dfrac{1}{x\ln(x+1)}$. Since f is continuous, positive, and decreasing on $[1, \infty)$, we can use the integral test. Note first that for all $x \in [1, \infty)$

$$\frac{1}{x\ln(x+1)} > \frac{1}{(x+1)\ln(x+1)}$$

Therefore

$$\int_1^b \frac{1}{x\ln(x+1)}dx > \int_1^b \frac{1}{(x+1)\ln(x+1)}dx = [\ln[\ln(x+1)]]_1^b$$
$$= \ln[\ln(b+1)] - \ln[\ln 2]$$

As $b \to \infty, \ln[\ln(b+1)] \to \infty$. *This shows that*

$$\int_1^\infty \frac{1}{x\ln(x+1)}dx$$

diverges. Therefore the series diverges.

1.4.7 Framing the Remainder of a Series

The following result gives a very simple framework for the remainder of a series whose general term is of the form $u_n = f(n)$.

Proposition 1.4.16 *The remainder* R_n *of a convergence series with general term* $u_n = f(n)$ *where* f *is positive, continuous and decreasing, verifies the following framework:*

$$\int_{n+1}^{+\infty} f(x)dx \leq R_n \leq \int_n^{+\infty} f(x)dx \ \forall n \in \mathbb{N},$$

where $R_n = \sum_{k=n+1}^{+\infty} u_k$.

Proof: It is sufficient to use the frames that follow from the proof of the theorem 1.4.12:

$$\int_k^{k+1} f(x)dx \leq u_k \leq \int_{k-1}^{k} f(x)dx,$$

and sum them for k from $n+1$ to infinity. □

1.4.8 Riemann Series

The comparison theorem (theorem 1.4.1) and the equivalents theorem (theorem 1.4.8) reduce the study of positive-term series to a catalog of series whose convergence is known. This catalog includes Riemann and Bertrand series. Let's start with the Riemann series $\sum_{k\geqslant 1} \frac{1}{k^\alpha}$, for $\alpha > 0$ a real number.

Proposition 1.4.17

$$\textit{Si} \quad \alpha > 1 \quad \textit{then} \sum_{k=1}^{+\infty} \frac{1}{k^\alpha} \quad \textit{converges}$$
$$\textit{If } 0 < \alpha \leqslant 1 \quad \textit{then} \sum_{k\geqslant 1} \frac{1}{k^\alpha} \quad \textit{diverge}$$

Proof: In the theorem 1.4.12, nothing obliges to start from 0 : for $m \in \mathbb{N}$, the series $\sum_{k\geqslant m} f(k)$ and the improper integral $\int_m^{+\infty} f(t)\mathrm{d}t$ are of the same nature.

We apply this to $f : [1, +\infty[\to [0, +\infty[$. defined by $f(t) = \dfrac{1}{t^\alpha}$. For $\alpha > 0$, it is a decreasing and positive function. We can apply the 1.4.12 theorem. We know that :

$$\int_1^x \frac{1}{t^\alpha} \mathrm{d}t = \begin{cases} \dfrac{1}{1-\alpha}\left(x^{1-\alpha} - 1\right) & \text{if } \alpha \neq 1 \\ \ln(x) & \text{if } \alpha = 1 \end{cases}$$

For $\alpha > 1$, $\displaystyle\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t$ is convergent, so the series $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^\alpha}$ converge.

For $0 < \alpha \leqslant 1$, $\displaystyle\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t$ is divergent, so the series $\displaystyle\sum_{k \geqslant 1} \frac{1}{k^\alpha}$ diverges. □

1.4.9 Bertrand Series

A more sophisticated family of series are Bertrand series: $\displaystyle\sum_{k \geqslant 2} \frac{1}{k^\alpha (\ln k)^\beta}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.

Proposition 1.4.18 *Let Bertrand's series*

$$\sum_{k \geqslant 2} \frac{1}{k^\alpha (\ln k)^\beta}$$

If $\alpha > 1$ *then it converges.*
If $0 < \alpha < 1$ *then it diverges.*

$$\textit{If} \quad \alpha = 1 \textit{ and } \left\{ \begin{array}{l} \beta > 1 \textit{ then it converges.} \\ \\ \beta \leqslant 1 \textit{ then it diverges.} \end{array} \right\}.$$

Proof: The demonstration is the same as for Riemann series. For example, for the case $\alpha = 1$:

$$\int_2^x \frac{1}{t(\ln t)^\beta} \mathrm{d}t = \begin{cases} \dfrac{1}{1-\beta}\left((\ln x)^{1-\beta} - (\ln 2)^{1-\beta}\right) & \text{si } \beta \neq 1 \\ \ln(\ln x) - \ln(\ln 2) & \text{si } \beta = 1 \end{cases}$$

□

1.4.10 Applications

In particular, we find that :

1. $\sum \dfrac{1}{k^2}$ converges (take $\alpha = 2$), while $\sum \dfrac{1}{k}$ diverges (take $\alpha = 1$).

Let's finish with two examples of using equivalents with Riemann and Bertrand series.

Example 1.4.19

1. *The serie*

$$\sum_{k \geqslant 1} \ln\left(1 + \frac{1}{\sqrt{k^3}}\right)$$

converges? As

$$\ln\left(1 + \frac{1}{\sqrt{k^3}}\right) \sim \frac{1}{\sqrt{k^3}}$$

and the Riemann series $\sum \frac{1}{\sqrt{k^3}} = \sum \frac{1}{k^{\frac{3}{2}}}$ *converges (because* $\frac{3}{2} > 1$ *), then by the equivalence theorem the series* $\sum_{k=1}^{+\infty} \ln\left(1 + \frac{1}{\sqrt{k^3}}\right)$ *also converges.*

2. *Is the series convergent?*

$$\sum_{k \geqslant 1} \frac{1 - \cos\left(\frac{1}{k\sqrt{\ln k}}\right)}{\sin\left(\frac{1}{k}\right)}$$

We are looking for an equivalent of the general term (which is positive):

$$\frac{1 - \cos\left(\frac{1}{k\sqrt{\ln k}}\right)}{\sin\left(\frac{1}{k}\right)} \sim \frac{1}{2k \ln k}$$

Bertrand's series $\sum \frac{1}{k \ln k}$ *diverges, so our series diverges too.*

1.4.11 Cauchy Roots Rule

Theorem 1.4.20 (Cauchy root test – General case) *Let* $\sum_{k=0}^{\infty} u_k$ *be a series of real (or complex) numbers.*

(i) If there exist a constant $0 < q < 1$ *and an integer* k_0 *such that, for all* $k \geqslant k_0$,

$$\sqrt[k]{|u_k|} \leqslant q < 1,$$

then $\sum u_k$ *converges absolutely.*

(ii) If there exists an integer k_0 *such that, for all* $k \geqslant k_0$,

$$\sqrt[k]{|u_k|} \geqslant 1,$$

then $\sum u_k$ *diverges (since* $\lim_{k\to\infty} u_k \neq 0$*).*

Most of the time, the test is applied to series with non-negative terms.

Proof: [Proof of Theorem 1.4.20] We prove each part separately.

Part (i): Assume there exist $0 < q < 1$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\sqrt[k]{|u_k|} \leq q.$$

Raising both sides to the power k gives:

$$|u_k| \leq q^k \quad \text{for all } k \geq k_0.$$

Consider the geometric series $\sum_{k=0}^{\infty} q^k$. Since $0 < q < 1$, this series converges (its sum is $\frac{1}{1-q}$).

By the comparison test for series with non-negative terms (applied to $\sum |u_k|$), the series $\sum_{k=0}^{\infty} |u_k|$ converges.

Therefore, $\sum_{k=0}^{\infty} u_k$ converges absolutely, hence it converges.

Part (ii): Assume there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\sqrt[k]{|u_k|} \geq 1.$$

Raising both sides to the power k yields:

$$|u_k| \geq 1 \quad \text{for all } k \geq k_0.$$

In particular, for $k \geq k_0$, we have $u_k \neq 0$. More importantly, the sequence (u_k) cannot converge to 0 because:

$$\liminf_{k\to\infty} |u_k| \geq 1 > 0.$$

By the divergence test (also called the n-th term test), if a series $\sum u_k$ converges, then $\lim_{k\to\infty} u_k = 0$. Since this condition fails, the series $\sum_{k=0}^{\infty} u_k$ diverges.

Remark about the boundary case: The theorem does not cover the case where $\limsup_{k\to\infty} \sqrt[k]{|u_k|} = 1$. In this situation, the series may converge or diverge. For example:

- For $u_k = \frac{1}{k}$, we have $\sqrt[k]{|u_k|} = \left(\frac{1}{k}\right)^{1/k} \to 1$, but $\sum \frac{1}{k}$ diverges (harmonic series).
- For $u_k = \frac{1}{k^2}$, we have $\sqrt[k]{|u_k|} = \left(\frac{1}{k^2}\right)^{1/k} \to 1$, but $\sum \frac{1}{k^2}$ converges.

This shows why the strict inequalities $q < 1$ and ≥ 1 (rather than ≤ 1) are essential in the statement. □

Corollary 1.4.21 (Cauchy root test for series with non-negative terms) *Let $\sum_{k=0}^{\infty} u_k$ be a series with non-negative terms ($u_k \geq 0$), and suppose that $\sqrt[k]{u_k}$ converges to ℓ.*

(i) If $\ell < 1$, then $\sum u_k$ converges.

(ii) If $\ell > 1$, then $\sum u_k$ diverges.

(iii) If $\ell = 1$, the test is inconclusive (the series may converge or diverge).

In practice, to compute the k-th root, we often use the exponential form:

$$\sqrt[k]{u_k} = (u_k)^{1/k} = \exp\left(\frac{1}{k} \ln u_k\right).$$

Proof: Recall that the convergence of a series is not affected by its first finitely many terms.

(i) In the first case of Theorem 1.4.20, $\sqrt[k]{|u_k|} \leqslant q$ implies $|u_k| \leqslant q^k$ for all $k \geq k_0$. Since $0 < q < 1$, the geometric series $\sum_{k=0}^{\infty} q^k$ converges. By the comparison test (for series with non-negative terms), $\sum |u_k|$ converges, hence $\sum u_k$ converges absolutely.

(ii) In the second case, $\sqrt[k]{|u_k|} \geqslant 1$ implies $|u_k| \geqslant 1$ for all $k \geq k_0$. Consequently, $\lim_{k\to\infty} u_k \neq 0$, and therefore the series $\sum u_k$ diverges by the divergence test.

(iii) For the last point of the corollary, consider the counterexamples:

$$u_k = \frac{1}{k} \quad \text{and} \quad v_k = \frac{1}{k^2}.$$

For both sequences, we have:

$$\sqrt[k]{u_k} = \exp\left(\frac{1}{k}\ln\left(\frac{1}{k}\right)\right) = \exp\left(-\frac{\ln k}{k}\right) \to e^0 = 1,$$

and similarly $\sqrt[k]{v_k} \to 1$. However, the harmonic series $\sum \frac{1}{k}$ diverges, while the series $\sum \frac{1}{k^2}$ converges (by the p-series test with $p = 2 > 1$). This shows that the case $\ell = 1$ yields no conclusion.

□

Example 1.4.22 *For example, the series,*

$$\sum \left(\frac{2k+1}{3k+4}\right)^k \qquad \textit{converges,}$$

car $\sqrt[k]{u_k} = \dfrac{2k+1}{3k+4}$ *tends to* $\dfrac{2}{3} < 1$. *On the other hand, whatever* $\alpha > 0$ *is*

$$\sum \frac{2^k}{k^\alpha} \ \textit{diverge,}$$

Indeed,

$$\sqrt[k]{u_k} = \frac{\sqrt[k]{2^k}}{(\sqrt[k]{k})^\alpha} = \frac{2}{\left(k^{\frac{1}{k}}\right)^\alpha} = \frac{2}{\left(\exp\left(\frac{1}{k}\ln k\right)\right)^\alpha} \to 2 > 1.$$

1.4.12 D'Alembert's Quotient Rule

d'Alembert's quotient rule is an efficient way of showing whether a series of real numbers converges or not.

Theorem 1.4.23 (D'Alembert's quotient rule - General case) *Let* $\sum u_k$ *be a series whose general terms are non-zero real numbers.*

1. *If there exists a constant* $0 < q < 1$ *and an integer* k_0 *such that, for any* $k \geqslant k_0$,

$$\left|\frac{u_{k+1}}{u_k}\right| \leqslant q < 1, \qquad \textit{then } \sum u_k \textit{ converges.}$$

The series is even absolutely convergent.

2. *If there exists an integer* k_0 *such that, for any* $k \geqslant k_0$,

$$\left|\frac{u_{k+1}}{u_k}\right| \geqslant 1, \qquad \textit{then } \sum u_k \textit{ diverge.}$$

The situation most often studied is when the series $\dfrac{u_{k+1}}{u_k}$ converges; the position of the limit with respect to 1 then determines the nature of the series. Here's a direct and most frequently used application, for series of real, strictly positive numbers:

Corollary 1.4.24 (D'Alembert's rule for series of real, strictly positive numbers) *Let* $\sum u_k$ *be a series with strictly positive terms, such that* $\dfrac{u_{k+1}}{u_k}$ *converges to* ℓ.

1. *If* $\ell < 1$ *then* $\sum u_k$ *converges.*
2. *If* $\ell > 1$ *then* $\sum u_k$ *diverges.*
3. *If* $\ell = 1$ *we can't conclude in general.*

Proof: Recall first that the geometric series $\sum q^k$ converges si $|q| < 1$, diverges otherwise. In the first case of the theorem, the assumption $\left|\frac{u_{k+1}}{u_k}\right| \leqslant q$ implies $|u_{k_0+1}| \leqslant |u_{k_0}|\, q$, then $|u_{k_0+2}| \leqslant |u_{k_0}|\, q^2$. We check by recurrence that, for any $k \geqslant k_0$:

$$|u_k| \leqslant |u_{k_0}|\, q^{-k_0} \cdot q^k = c \cdot q^k$$

where c is a constant. Since $0 < q < 1$, then the series $\sum q^k$ converges, Where the result by theorem 1.4.3: the series $\sum |u_k|$ converges. If $\left|\frac{u_{k+1}}{u_k}\right| \geqslant 1$, the sequence $(|u_k|)$ is increasing: it cannot therefore tend towards 0 and the series diverges. □

Remark 1.4.25 *The theorem cannot be applied if some u_k is zero, contrary to the Cauchy root rule we'll see later.*

- *Note that the theorem doesn't always lead to a conclusion. Be careful that the hypothesis is $\left|\frac{u_{k+1}}{u_k}\right| \leqslant q < 1$, which is stronger than $\left|\frac{u_{k+1}}{u_k}\right| < 1$.*
- *Similarly, the corollary does not allow us to conclude when $\frac{u_{k+1}}{u_k} \to 1$. For example, for the series $\sum u_k = \sum \frac{1}{k}$ and $\sum v_k = \sum \frac{1}{k^2}$ we have $\frac{u_{k+1}}{u_k} = \frac{k}{k+1} \to 1$, and $\frac{v_{k+1}}{v_k} = \frac{k^2}{(k+1)^2}. \to 1$. However, the series $\sum \frac{1}{k}$ diverges while $\sum \frac{1}{k^2}$ converges. If the limit of the sequence $(\frac{u_{n+1}}{u_n})_n$ does not exist, it may be that the series $\sum u_n$ converges or diverges. Consider the following sequences:*

  1. *$\sum u_n = (2+(-1)^n)2^{-n}$, here, the limit of $(\frac{u_{n+1}}{u_n})_n$ doesn't exist and $\sum u_n$ converges since it's the sum of two convergent geometric series.*
  2. *$u_n = (2+(-1)^n)$, in this case the limit doesn't exist and $\sum u_n$ diverges since its general term doesn't tend towards 0.*

1.4.13 D'Alembert vs Cauchy

Let's compare D'Alembert's quotient rule with Cauchy's root rule. We'll see that Cauchy's root rule is more powerful than d'Alembert's quotient rule. In practice, however, d'Alembert's quotient rule is still the most widely used.

Proposition 1.4.26 *Let (u_k) be a sequence with strictly positive terms.*

$$\text{If} \quad \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \ell \quad \text{then} \quad \lim_{k\to+\infty} \sqrt[k]{u_k} = \ell.$$

In other words, if we can apply D'Alembert's quotient rule, then we can also apply Cauchy's root rule.

Proof: For any $\varepsilon > 0$, there exists k_0 such that, for any $k \geqslant k_0$,

$$\ell - \varepsilon < \frac{u_{k+1}}{u_k} < \ell + \varepsilon.$$

By recurrence, we deduce:

$$u_{k_0}(\ell - \varepsilon)^{k-k_0} \leqslant u_k \leqslant u_{k_0}(\ell + \varepsilon)^{k-k_0}$$

Or :

$$\lim_{k\to+\infty} \sqrt[k]{u_{k_0}(\ell-\varepsilon)^{k-k_0}} = \ell-\varepsilon \quad \text{and} \quad \lim_{k\to+\infty} \sqrt[k]{u_{k_0}(\ell+\varepsilon)^{k-k_0}} = \ell+\varepsilon.$$

So there exists $k_1 > k_0$ such that, for $k > k_1$,

$$\ell - 2\varepsilon < \sqrt[k]{u_k} < \ell + 2\varepsilon$$

where the result. □

Let's finish with an example where Cauchy's root rule allows us to conclude, but d'Alembert's quotient rule does not.

Example 1.4.27 *Let's define the sequence u_k by :*

$$u_k = \begin{cases} \dfrac{2^n}{3^n} & \text{if } k = 2n \\ \dfrac{2^n}{3^{n+1}} & \text{if } k = 2n+1 \end{cases}$$

The ratio $\dfrac{u_{k+1}}{u_k}$ is $\dfrac{1}{3}$ if k is even, 3 if k is odd. D'Alembert's quotient rule therefore does not apply. However, $\sqrt[k]{u_k}$ converges to $\sqrt{\dfrac{2}{3}} < 1$, so Cauchy's root rule applies and the series $\sum u_k$ converges.

1.4.14 Duhamel's Rule

D'Alembert's rule (respectively. Cauchy's rule) does not allow us to check the nature of the series $\sum_{n\geq 0} u_n$ when

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1.$$

This is an opportunity to use another technique to establish the nature of a series with positive terms. This technique is called Duhamel's rule and is given by the following theorem:

Theorem 1.4.28 *Let $\sum_{n\geq 0} u_n$ be a series with positive terms. It is assumed that :*

$$\frac{u_{n+1}}{u_n} = 1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right).$$

Then

1. *If $\lambda > 1$ the $\sum_{n\geq 0} u_n$ series converges.*
2. *If $\lambda < 1$ the series $\sum_{n\geq 0} u_n$ diverges.*

Proof:

1. If $\lambda > 1$, there exists α such that $\lambda > \alpha > 1$. We put

$$v_n = \frac{1}{n^\alpha}.$$

Then,

$$\frac{v_{n+1}}{v_n} = \left(\frac{n}{n+1}\right)^\alpha = \left(1+\frac{1}{n}\right)^{-\alpha}.$$

A limit development in the vicinity of infinity of order 1 gives

$$\left(1+\frac{1}{n}\right)^{-\alpha} = 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right).$$

So $-\frac{\lambda}{n} < -\frac{\alpha}{n}$. Therefore,

$$1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right) < 1 - \frac{\alpha}{n} + o\left(\frac{1}{n}\right).$$

i.e.

$$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}.$$

Where By applying the comparison theorem, we deduce that $\sum_{n\geq 0} u_n$ converges since the Riemann series $\sum_{n\geq 0} v_n = \sum_{n\geq 0} \frac{1}{n^\alpha}$ is convergent ($\alpha > 1$).

2. If $\lambda < 1$, we put

$$v_n = \frac{1}{n}.$$

Then,

$$\frac{v_{n+1}}{v_n} = \left(\frac{n}{n+1}\right) = \frac{1}{\left(1+\frac{1}{n}\right)}.$$

On the other hand, since $\lambda < 1$ then

$$1 - \frac{\lambda}{n} + o\left(\frac{1}{n}\right) > -\frac{1}{n} + 1 + o\left(\frac{1}{n}\right).$$

This is equivalent to

$$\frac{v_{n+1}}{v_n} \leq \frac{u_{n+1}}{u_n}.$$

And according to the comparison theorem $\sum_{n\geq 0} u_n$ diverges since the series $\sum_{n\geq 0} v_n$ is a divergent harmonic series.

□

Example 1.4.29 *Consider the series with general term u_n defined by :*

$$u_n = \frac{(2n)!}{2^{2n}(n!)^2}.$$

Then,

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} = 1 - \frac{1}{2n+2} = 1 - \frac{1}{2n}\left(\frac{1}{1+\frac{1}{2n}}\right). \tag{1.14}$$

In addition, a development limited to the vicinity of infinity of order 1 gives

$$\left(1+\frac{1}{2n}\right)^{-1} = 1-\frac{1}{2n}+o\left(\frac{1}{n}\right). \tag{1.15}$$

Applying (1.15) to (1.14), we find

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= 1-\frac{1}{2n}\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right),\\ &= 1-\frac{1}{2n}+\frac{1}{4n^2}+o\left(\frac{1}{n^2}\right).\end{aligned}$$

From Duhamel's theorem, we conclude that $\sum_{n\geq 0} u_n$ *diverges since* $\lambda = \frac{1}{2} < 1$.

1.4.15 Riemann's Rule or "$n^\alpha u_n$"

This rule is very useful in exercises.

Proposition 1.4.30 *Let* $\sum_{n\geq 1} u_n$ *be a series with positive terms and* α *be a real number.*

1. *If* $\alpha > 1$ *and the sequence* $(n^\alpha u_n)_n$ *tends to 0, then the series* $\sum_{n\geq 1} u_n$ *converges.*

2. *If* $\alpha \leq 1$ *and the series* $(n^\alpha u_n)_n$ *tends to* $+\infty$*, then the series* $\sum_{n\geq 1} u_n$ *diverges.*

Proof: The proof is immediate, and follows from comparing the series $\sum_{n\geq 1} u_n$ with the Riemann series $\sum_{n\geq 1} \frac{1}{n^\alpha}$. □

Example 1.4.31 *The infinite series* $\sum_{n\geq 1} e^{-2\sqrt{n}}$ *converges since we have* $\lim_{n\to+\infty} n^2 e^{-2\sqrt{n}} = 0$.

1.5 Alternating Series

There's another type of series that's easy to study: alternating series. These are series in which the sign of the general term changes at each rank.

1.5.1 Leibniz Criterion

Let $(u_k)_{k\geqslant 0}$ be a sequence that verifies $u_k \geqslant 0$. The series $\sum_{k\geqslant 0}(-1)^k u_k$ is called an alternating series. We have the following convergence criterion, which is extremely easy to verify:

Theorem 1.5.1 (Leibniz criterion) *Suppose that* $(u_k)_{k\geqslant 0}$ *is a sequence that verifies :*

1. $u_k \geqslant 0$ *for all* $k \geqslant 0$,

2. *the sequence* (u_k) *is a decreasing sequence,*

3. *and* $\lim_{k\to+\infty} u_k = 0$.

Then the alternating series $\sum_{k=0}^{+\infty}(-1)^k u_k$ *converges.*

Proof: Let's get back to two adjacent sequences.

- The sequence (S_{2n+1}) is increasing because $S_{2n+1} - S_{2n-1} = u_{2n} - u_{2n+1} \geqslant 0$.
- The sequence (S_{2n}) is decreasing because $S_{2n} - S_{2n-2} = u_{2n} - u_{2n-1} \leqslant 0$.
- $S_{2n} \geqslant S_{2n+1}$ because $S_{2n+1} - S_{2n} = -u_{2n+1} \leqslant 0$.
- Finally $S_{2n+1} - S_{2n}$ tends to 0 because $S_{2n+1} - S_{2n} = -u_{2n+1} \to 0$ (when $n \to +\infty$).

Consequently (S_{2n+1}) and (S_{2n}) converge, and converge to the same limit S. We conclude that (S_n) converges to S. In addition, we've shown that $S_{2n+1} \leqslant S \leqslant S_{2n}$ for all n. Finally, we also have and

$$0 \geqslant R_{2n} = S - S_{2n} \geqslant S_{2n+1} - S_{2n} = -u_{2n+1}$$

Thus, whatever the parity of n, we have $|R_n| = |S - S_n| \leqslant u_{n+1}$. □

Example 1.5.2 *The alternating harmonic series*

$$\sum_{k=0}^{+\infty} (-1)^k \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. Indeed, by posing $u_k = \dfrac{1}{k+1}$*, then*

1. $u_k \geqslant 0$,
2. (u_k) *is a decreasing sequence,*
3. *The sequence* (u_k) *converges to 0.*

By the Leibniz criterion (theorem 1.5.1), the alternating series $\sum_{k=0}^{+\infty} (-1)^k \frac{1}{k+1}$ *converges.*

1.5.2 The Remainder

Not only does the Leibniz criterion prove the convergence of the series $\sum_{k=0}^{+\infty} (-1)^k u_k$, but the proof provides us with two additional important results: a framing of the sum and a majorization of the remainder.

Corollary 1.5.3 *Let be an alternating series* $\sum_{k=0}^{+\infty} (-1)^k u_k$ *verifying the assumptions of the theorem 1.5.1. Let* S *be the sum of this series and let* (S_n) *be the sequence of partial sums.*

1. *The sum* S *satisfies the bounds:*

$$S_1 \leqslant S_3 \leqslant S_5 \leqslant \cdots \leqslant S_{2n+1} \leqslant \cdots \leqslant S \leqslant \cdots \leqslant S_{2n} \leqslant \cdots \leqslant S_4 \leqslant S_2 \leqslant S_0.$$

2. *Furthermore, if* $R_n = S - S_n = \sum_{k=n+1}^{+\infty} (-1)^k u_k$ *is the remainder of order* n*, then we have*

$$|R_n| \leqslant u_{n+1}.$$

For an alternating series, the speed of convergence is therefore dictated by the decay of the sequence (u_k) *towards 0. This can be quite slow.*

Example 1.5.4 *For example, we saw that the alternating harmonic series* $\sum_{k=0}^{+\infty} \frac{(-1)^k}{k+1}$ *converges; let's denote* S *its sum. The partial sums are* $S_0 = 1, S_1 = 1 - \frac{1}{2}, S_2 = 1 - \frac{1}{2} + \frac{1}{3}, S_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}, \ldots$ *The framework of the corollary is written as*

$$1-\frac{1}{2} \leqslant 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} \leqslant \cdots \leqslant S_{2n+1} \leqslant \cdots \leqslant S \leqslant \cdots \leqslant S_{2n} \leqslant \cdots \leqslant 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} \leqslant 1-\frac{1}{2}+\frac{1}{3} \leqslant 1$$

We deduce

$$S_3 = \frac{35}{60} \simeq 0.58333\ldots \leqslant S \leqslant S_4 = \frac{47}{60} \simeq 0.78333\ldots$$

If we push the calculations further, then for $n = 200$ *we obtain*

$$S_{201} \simeq 0.69067\ldots \leqslant S \leqslant S_{200} \simeq 0.69562\ldots$$

This gives us the first two decimal places of $S \simeq 0.69\ldots$. *In addition, we have an increase in the error committed, using the inequality* $|R_n| \leqslant u_{n+1}$. *We find that the error committed by approximating* S *by* S_{200} *is:* $|S - S_{200}| = |R_{200}| \leqslant u_{201} = \frac{1}{202} < 5 \cdot 10^{-3}$. *In fact, you will see later that* $S = \ln 2 \simeq 0.69314\ldots$

1.5.3 Counter-example

Let's end with two caveats:

1. We can't drop the decay condition of the sequence (u_k) in the Leibniz criterion.
2. It is not possible to replace u_k by an infinite equivalent in Theorem 1.5.1, as decay is not preserved by equivalence.

Example 1.5.5 *Here are two alternating series:*

$$\sum_{k \geqslant 2} \frac{(-1)^k}{\sqrt{k}} \quad \text{converges,} \qquad \sum_{k \geqslant 2} \frac{(-1)^k}{\sqrt{k} + (-1)^k} \quad \text{diverge.}$$

The Leibniz criterion (theorem 1.5.1) applies to the first: the sequence $u_k = \frac{1}{\sqrt{k}}$ *is a positive, decreasing sequence tending towards 0. Consequently, the alternating series* $\sum_{k \geqslant 2} \frac{(-1)^k}{\sqrt{k}}$ *converge. On the other hand, Leibniz's criterion does not apply to the second, because although the sequence* $v_k = \frac{1}{\sqrt{k} + (-1)^k}$ *is positive (for* $k \geqslant 2$ *) and tends towards 0 , it is not decreasing. However, we do have :*

$$v_k = \frac{1}{\sqrt{k} + (-1)^k} \sim \frac{1}{\sqrt{k}} = u_k$$

To show that $\sum_{k \geqslant 2} \frac{(-1)^k}{\sqrt{k} + (-1)^k}$ *diverges, let's calculate the difference:*

$$\begin{aligned} (-1)^k u_k - (-1)^k v_k &= \frac{(-1)^k}{\sqrt{k}} - \frac{(-1)^k}{\sqrt{k} + (-1)^k} = (-1)^k \frac{\sqrt{k} + (-1)^k - \sqrt{k}}{k + (-1)^k \sqrt{k}} \\ &= \frac{1}{k + (-1)^k \sqrt{k}} \sim \frac{1}{k} \end{aligned}$$

Thus the series with general term $w_k = (-1)^k u_k - (-1)^k v_k$ diverges, since its general term is equivalent to that of the harmonic series $\sum \frac{1}{k}$ which diverges. Let's now assume for the sake of argument that the $\sum_{k\geqslant 2}(-1)^k v_k$ series is convergent. We also know that the series $\sum_{k\geqslant 2}(-1)^k u_k$ is convergent. Therefore, by linearity, the series $\sum_{k\geqslant 2} w_k = \sum_{k\geqslant 2}(-1)^k u_k - \sum_{k\geqslant 2}(-1)^k v_k$ would be convergent. Which is a contradiction. Conclusion: the series $\sum_{k\geqslant 2} \frac{(-1)^k}{\sqrt{k}+(-1)^k}$ diverge.

1.6 Series with Terms of Arbitrary Signs

1.6.1 Absolutely Convergent Series

Definition 1.6.1 *A series $\sum_{k\geqslant 0} u_k$ of real numbers is said to be absolutely convergent if the series $\sum_{k\geqslant 0} |u_k|$ is convergent.*

Example 1.6.2 *1. The series $\sum_{k\geqslant 1} \frac{\cos k}{k^2}$ is absolutely convergent. Because for $u_k = \frac{\cos k}{k^2}$ we have $|u_k| \leqslant \frac{1}{k^2}$. Since the series $\sum_{k\geqslant 1} \frac{1}{k^2}$ converges then $\sum_{k\geqslant 1} |u_k|$ also converges.*

2. The alternating harmonic series $\sum_{k=0}^{+\infty} \frac{(-1)^k}{k+1}$ is not absolutely convergent. Because for $v_k = \frac{(-1)^k}{k+1}$, the series $\sum_{k\geqslant 0} |v_k| = \sum_{k\geqslant 0} \frac{1}{k+1}$ diverges.

A series, such as the alternating harmonic series, which is convergent, but not absolutely convergent, is called a conditionally convergent series.
Being absolutely convergent is stronger than being convergent:

Theorem 1.6.3 *Any absolutely convergent series is convergent.*

Proof: Let's use the Cauchy criterion. Let $\sum u_k$ be an absolutely convergent series. The series $\sum |u_k|$ is convergent, so the sequence of remainders (R'_n) with $R'_n = \sum_{k=n+1}^{+\infty} |u_k|$ is a sequence that tends to 0 , so in particular it is a Cauchy sequence. Let $\varepsilon > 0$ be fixed. There therefore exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and for all $p \geqslant 0$:

$$|u_n| + |u_{n+1}| + \cdots + |u_{n+p}| < \varepsilon$$

Therefore, for $n \geqslant n_0$ and $p \geqslant 0$ we have:

$$|u_n + u_{n+1} + \cdots + u_{n+p}| \leqslant |u_n| + |u_{n+1}| + \cdots + |u_{n+p}| < \varepsilon.$$

So, according to the Cauchy criterion (Theorem 1.2.18), $\sum u_k$ is convergent. □

Remark 1.6.4 *The converse of Theorem 1.6.3 is false i.e.*

$$\sum_{n\geq 0} u_n \text{ converges } \not\Rightarrow \sum_{n\geq 0} |u_n| \text{ converges }.$$

Example 1.6.5 *The Leibniz series $\sum_{n\geq 0} \frac{(-1)^n}{2n}$ converges without being absolutely convergent.*

1.6.2 Converge Conditionally

Definition 1.6.6 *A series* $\sum_{n\geq 0} u_n$ *is said to* ***converge conditionally*** *if* $\sum_{n\geq 0} u_n$ *converges, but not absolutely. A conditionally convergent series is any convergent series that does not converge absolutely.*

Example 1.6.7 *Consider the alternating Riemann series*

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^\alpha} \quad \text{where } \alpha > 0.$$

This series is convergent by applying Leibniz's Theorem. Therefore, it is conditionally convergent if $0 < \alpha \leq 1$.

1.6.3 Cauchy and D'Alembert's Rule for Series with Terms of Arbitrary Signs

These two rules (see Theorems **??** and 1.4.23) are a priori applicable to series with positive terms. If we have a series $\sum u_n$ with terms of arbitrary signs, we can look at the absolute convergence of the series by using one or the other of these two rules. More precisely, we apply these two rules to the series $\sum |u_n|$.

Example 1.6.8

1. *For all* $x \in \mathbb{R}$ *fixed, the exponential series*

$$\sum_{k=0}^{+\infty} \frac{x^k}{k!} \quad \text{converges.}$$

Indeed, for $u_k = \dfrac{x^k}{k!}$ *we have*

$$\left|\frac{u_{k+1}}{u_k}\right| = \frac{\left|\dfrac{x^{k+1}}{(k+1)!}\right|}{\left|\dfrac{x^k}{k!}\right|} = \frac{|x|}{k+1} \to 0 \qquad \text{as } k \to +\infty$$

The limit being $= 0 < 1$ *then by D'Alembert's Ratio Test, the series is absolutely convergent, therefore convergent. By definition, the sum is* $\exp(x)$*:*

$$\exp(x) = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$$

2. $\sum_{k\geqslant 0} \dfrac{k!}{1\cdot 3\cdots(2k-1)}$ *converges, because* $\dfrac{u_{k+1}}{u_k} = \dfrac{k+1}{2k+1}$ *tends to* $\dfrac{1}{2} < 1$.

3. $\sum_{k\geqslant 0} \dfrac{(2k)!}{(k!)^2}$ *diverges, because* $\dfrac{u_{k+1}}{u_k} = \dfrac{(2k+1)(2k+2)}{(k+1)^2}$ *tends to* $4 > 1$.

Let's finish with a more complicated example.

Example 1.6.9 *Find all $z \in \mathbb{C}$ such that the series $\sum_{k \geqslant 0} \begin{pmatrix} k \\ \\ 3 \end{pmatrix} z^k$ is absolutely convergent.*

Let $u_k = \begin{pmatrix} k \\ \\ 3 \end{pmatrix} z^k$. *Then, for* $z \neq 0$,

$$\frac{|u_{k+1}|}{|u_k|} = \frac{\begin{pmatrix} k+1 \\ \\ 3 \end{pmatrix} |z|^{k+1}}{\begin{pmatrix} k \\ \\ 3 \end{pmatrix} |z|^k} = \frac{\frac{(k+1)k(k-1)}{3!}}{\frac{k(k-1)(k-2)}{3!}} |z| = \frac{k+1}{k-2}|z| \to |z| \qquad \text{as } k \to +\infty.$$

If $|z| < 1$ *then for* k *large enough* $\frac{|u_{k+1}|}{|u_k|} < q < 1$ *so the series* $\sum u_k$ *is absolutely convergent.*
If $|z| \geqslant 1$ *then* $\frac{|u_{k+1}|}{|u_k|} = \frac{k+1}{k-2}|z| \geqslant \frac{k+1}{k-2} > 1$ *for all* k. *Therefore, the series* $\sum u_k$ *diverges.*

Example 1.6.10 *Determine all $z \in \mathbb{C}$ such that the series $\sum_{k \geqslant 1} \left(1 + \frac{1}{k}\right)^{k^2} z^k$ is absolutely convergent.*
Let $u_k = \left(1 + \frac{1}{k}\right)^{k^2} z^k$. *We have*

$$\sqrt[k]{|u_k|} = \left(1 + \frac{1}{k}\right)^k |z| \to e|z|.$$

This limit satisfies $e|z| < 1$ *if and only if* $|z| < \frac{1}{e}$.

- *If* $|z| < \frac{1}{e}$ *then the series* $\sum u_k$ *is absolutely convergent.*
- *If* $|z| > \frac{1}{e}$, *we have for* k *large enough* $\sqrt[k]{|u_k|} > 1$, *so the series* $\sum u_k$ *diverges.*
- *If* $|z| = \frac{1}{e}$ *the Cauchy Root Test does not allow us to conclude. We study the general term by hand. We obtain:*

Therefore

$$|u_k| = \left(1 + \frac{1}{k}\right)^{k^2} \left(\frac{1}{e}\right)^k$$

$$\begin{aligned}
\ln |u_k| &= k^2 \ln\left(1+\frac{1}{k}\right) + k \ln \frac{1}{e} \\
&= k\left[k \ln\left(1+\frac{1}{k}\right) - 1\right] \\
&= k\left[k\left(\frac{1}{k} - \frac{1}{2}\left(\frac{1}{k}\right)^2 + o\left(\frac{1}{k^2}\right)\right) - 1\right] \\
&= k\left[1 - \frac{1}{2}\frac{1}{k} + o\left(\frac{1}{k}\right) - 1\right] \\
&= -\frac{1}{2} + o(1) \\
&\to -\frac{1}{2}
\end{aligned}$$

So $|u_k| \to e^{-\frac{1}{2}} \neq 0$*. Thus* $\sum |u_k|$ *diverges.*

1.6.4 Raabe-Duhamel's Rule

D'Alembert's Ratio Test and Cauchy's Root Test do not apply to the Riemann series

$$\sum_{k \geqslant 1} \frac{1}{k^\alpha}$$

car $\dfrac{k^\alpha}{(k+1)^\alpha} \to 1$ and $\sqrt[k]{u_k} \to 1$. We need to refine D'Alembert's Rule to be able to conclude. However, we will return to the convergence of Riemann series using other techniques.

Theorem 1.6.11 (Raabe-Duhamel's Rule) *Let* (u_k) *be a sequence of non-zero real (or complex) numbers.*

1. *If* $\forall k \geqslant k_0$ *we have* $\left|\dfrac{u_{k+1}}{u_k}\right| \leqslant 1 - \dfrac{\beta}{k}$*, with* $\beta > 1$*, then the series* $\sum u_k$ *is absolutely convergent.*
2. *If* $\forall k \geqslant k_0$ *we a* $\left|\dfrac{u_{k+1}}{u_k}\right| \geqslant 1 - \dfrac{1}{k}$*, then the series* $\sum u_k$ *is not absolutely convergent.*

Caution! There are convergent series, although $\left|\dfrac{u_{k+1}}{u_k}\right| \geqslant 1 - \dfrac{1}{k}$. By the second point, such a series cannot be absolutely convergent.
Indeed, let $u_k = (-1)^k \dfrac{1}{k}$. Then:

$$\frac{|u_{k+1}|}{|u_k|} = \frac{k}{k+1} = 1 - \frac{1}{k+1} \geqslant 1 - \frac{1}{k}$$

Proof:

1. The assumption implies $k\,|u_{k+1}| \leqslant k\,|u_k| - \beta\,|u_k|$ (for all $k \geqslant k_0$).
   Thus
   $$(\beta - 1)\,|u_k| \leqslant (k-1)\,|u_k| - k\,|u_{k+1}|\,.$$
   Since $\beta > 1$ then the above inequality implies $(k-1)\,|u_k| - k\,|u_{k+1}| > 0$ and thus $(k-1)\,|u_k| > k\,|u_{k+1}|$. The sequence $(k\,|u_{k+1}|)_{k \geqslant k_0}$ is decreasing and bounded below by 0 ;

this sequence therefore has a limit. Thus, the telescoping series $\sum [(k-1)|u_k| - k|u_{k+1}|]$ converges. Since

$$(\beta - 1)|u_k| \leqslant (k-1)|u_k| - k|u_{k+1}|,$$

the series $\sum(\beta - 1)|u_k|$ converges and therefore also $\sum |u_k|$.

2. The assumption implies $k|u_{k+1}| \geqslant (k-1)|u_k| > 0$ (for all $k \geqslant k_0$). Therefore, the sequence $(k|u_{k+1}|)_{k \geqslant k_0}$ is increasing, so $k|u_{k+1}| \geqslant \varepsilon > 0$. Therefore, for all $k \geqslant k_0$, we have $|u_{k+1}| \geqslant \dfrac{\varepsilon}{k}$. Therefore, $\sum |u_k|$ diverges, because $\sum \dfrac{1}{k}$ diverges.

□

1.7 Products of Two Series

For a product of sums, there are several ways of ordering the terms once the product has been developed. In the case of a finite sum, the order of the terms is unimportant, but in the case of a series it's essential. We choose to group the terms according to their indices, as follows:

$$(a_0 + a_1)(b_0 + b_1) = \underbrace{a_0 b_0}_{\text{sum of indices }=0} + \underbrace{a_0 b_1 + a_1 b_0}_{\text{sum of indices }=1} + \underbrace{a_1 b_1}_{\text{sum of indices }=2}$$

$$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = \underbrace{a_0 b_0}_{\text{sum of indices }=0} + \underbrace{a_0 b_1 + a_1 b_0}_{\text{sum of indices }=1}$$

$$+ \underbrace{a_0 b_2 + a_1 b_1 + a_2 b_0}_{\text{sum of indices }=2} + \underbrace{a_1 b_2 + a_2 b_1}_{\text{sum of indices }=3} + \underbrace{a_2 b_2}_{\text{sum of indices }=4}$$

More generally, here are different ways of writing a product of two sums:

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{j=0}^{n} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i b_j = \sum_{0 \leqslant k \leqslant 2n}\ \sum_{i+j=k} a_i b_j = \sum_{0 \leqslant k \leqslant 2n}\ \sum_{0 \leqslant i \leqslant k} a_i b_{k-i}.$$

The last two forms correspond to our decomposition according to the sum of indices.

1.7.1 Cauchy Product

Definition 1.7.1 *Let* $\sum_{i \geqslant 0} u_i$ *and* $\sum_{j \geqslant 0} v_j$ *be two series. The series* $\sum_{k \geqslant 0} w_k$ *where*

$$w_k = \sum_{i=0}^{k} u_i v_{k-i} \tag{1.16}$$

Another way of writing the coefficient w_k *is :*

$$w_k = \sum_{i+j=k} u_i v_j$$

From formula (1.16), we have

$$w_0 = u_0 v_0,\ w_1 = u_0 v_1 + u_1 v_0 \text{ and } w_n = u_0 v_n + u_1 v_{n-1} + \dots + u_{n-1} v_1 + u_n v_0.$$

Theorem 1.7.2 *If the series* $\sum_{i=0}^{+\infty} u_i$ *and* $\sum_{j=0}^{+\infty} v_j$ *of real numbers are absolutely convergent, then the product series*

$$\sum_{k=0}^{+\infty} w_k = \sum_{k=0}^{+\infty}\left(\sum_{i=0}^{k} u_i v_{k-i}\right)$$

is absolutely convergent and we have :

$$\sum_{k=0}^{+\infty} w_k = \left(\sum_{i=0}^{+\infty} u_i\right) \times \left(\sum_{j=0}^{+\infty} v_j\right).$$

Proof: Notations

- $S_n = u_0 + \cdots + u_n, S_n \to S,$
- $T_n = v_0 + \cdots + v_n, T_n \to T,$
- $P_n = w_0 + \cdots + w_n.$

We need to show that $P_n \to S \cdot T$.

First case: $u_k \geqslant 0, v_k \geqslant 0$ $(\forall k)$.

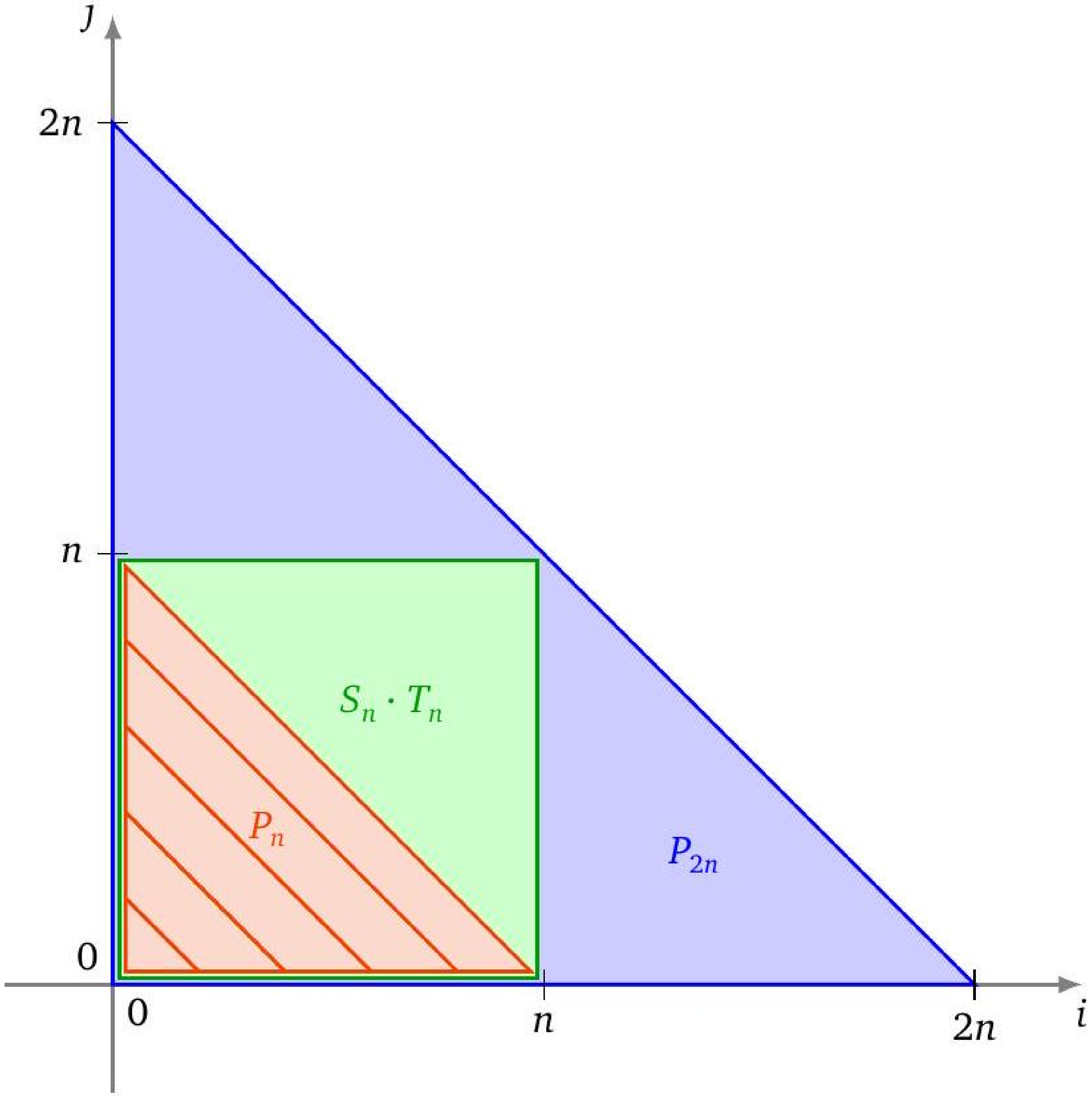
In this case $w_k \geqslant 0$ and we have

$$P_n \leqslant S_n \cdot T_n \leqslant S \cdot T.$$

The sequence (P_n) is increasing and major, and therefore convergent: $P_n \to P$.

We also have

$$P_n \leqslant S_n \cdot T_n \leqslant P_{2n}$$



The drawing represents the point corresponding to the indices (i, j). The red triangle represents P_n (with the grouping of terms corresponding to the diagonals), the green square corresponds to the product $S_n \cdot T_n$, the blue triangle represents P_{2n}. The fact that the square lies between the two triangles reflects the double inequality $P_n \leqslant S_n \cdot T_n \leqslant P_{2n}$.

So by making $n \to +\infty$, we have : $P \leqslant S \cdot T \leqslant P$. So $P_n \to S \cdot T$.

Second case: $u_k \in \mathbb{R}, v_k \in \mathbb{R}(\forall k)$.

We put :

- $S_n' = |u_0| + \cdots + |u_n|, S_n' \to S'$,
- $T_n' = |v_0| + \cdots + |v_n|, T_n' \to T'$, - $P_n' = w_0' + \cdots + w_n'$ where $w_k' = \sum_{i=0}^{k} |u_i v_{k-i}|$.

From the first case, $P_n' \to P'$ with $P' = S' \cdot T'$. Thus

$$|S_n \cdot T_n - P_n| = \left| \sum_{\substack{0 \leq i,j \leq n \\ i+j>n}} u_i v_j \right| \leqslant \sum_{\substack{0 \leq i,j \leq n \\ i+j>n}} |u_i v_j| = S'_n \cdot T'_n - P'_n \to S' \cdot T' - P' = 0.$$

Thus $P_n = S_n \cdot T_n - (S_n \cdot T_n - P_n) \to S \cdot T - 0 = S \cdot T$. So the series $\sum c_k$ is convergent and its sum is $S \cdot T$. Moreover, $|w_k| \leqslant w'_k$. Convergence of $\sum w'_k$ therefore implies absolute convergence of $\sum w_k$. □

Example 1.7.3 *Let* $\sum_{i=0}^{+\infty} u_i$ *be an absolutely convergent series and let* $\sum_{j=0}^{+\infty} v_j$ *be the series defined by* $v_j = \frac{1}{2^j}$. *The series* $\sum v_j$ *is absolutely convergent.*

Note

$$w_k = \sum_{i=0}^{k} u_i v_{k-i} = \sum_{i=0}^{k} u_i \times \frac{1}{2^{k-i}}.$$

Then the series $\sum w_k$ *converges absolutely and*

$$\sum_{k=0}^{+\infty} w_k = \left(\sum_{i=0}^{+\infty} u_i \right) \times \left(\sum_{j=0}^{+\infty} v_j \right) = 2 \sum_{i=0}^{+\infty} u_i.$$

1.7.2 Counter-example

If the series $\sum u_i$ and $\sum v_j$ are not absolutely convergent, but only convergent, then the Cauchy series can be divergent.

Example 1.7.4 *Let* $u_i = v_i = \frac{(-1)^i}{\sqrt{i+1}}, i \geqslant 0$. *Then* $\sum u_i$ *and* $\sum v_j$ *are convergent by the Leibniz criterion, but are not absolutely convergent. We have*

$$w_k = \sum_{i=0}^{k} u_i v_{k-i} = \sum_{i=0}^{k} \frac{(-1)^i}{\sqrt{i+1}} \frac{(-1)^{k-i}}{\sqrt{k-i+1}} = (-1)^k \sum_{i=0}^{k} \frac{1}{\sqrt{(i+1)(k-i+1)}}$$

Now, for $x \in \mathbb{R}, (x+1)(k-x+1) = -x^2 + kx + (k+1) \leq \frac{(k+2)^2}{4}$ *(value at vertex of parabola). From where* $\sqrt{(i+1)(k-i+1)} \leqslant \frac{(k+2)}{2}$. *Thus*

$$|w_k| = \sum_{i=0}^{k} \frac{1}{\sqrt{(i+1)(k-i+1)}} \geqslant \sum_{i=0}^{k} \frac{2}{k+2} = \frac{2(k+1)}{k+2} \to 2.$$

So the general term w_k *cannot tend 0, so the series* $\sum w_k$ *diverges.*

1.8 Grouping of Terms

In general, the series obtained by grouping the terms of a given series can converge without converging the initial series: the partial sums of the series obtained after grouping only form a sequence extracted from the sequence of partial sums of the initial series. Consider the series

$$\sum_{n\in\mathbb{N}} u_n = u_0 + u_1 + ... + u_n .../u_n \in \mathbb{R}\forall n \in \mathbb{N}.$$

We group the terms of the $\sum_{n\in\mathbb{N}} u_n$ series with conservation of order. We then have

$$\underbrace{(u_0 + u_1 + ... + u_{n_1})}_{v_0} + \underbrace{(u_{n_1+1} + u_{n_1+2} + ... + u_{n_2})}_{v_1} + ...$$
$$+ \underbrace{(u_{n_k+1} + ... + u_{n_{(k+1)}})}_{v_k}.$$

In other words, we have obtained the series :

$$\sum_{n\geq 0} v_n \text{ where } v_n = \sum_{p=\varphi(n)}^{\varphi(n+1)-1} u_p, \text{ et } \varphi(0) = 0, \ \varphi(k) = n_k + 1, \forall k \geq 1.$$

Definition 1.8.1 *We say that the series* $\sum_{n\geq 0} v_n$ *is deduced from the series* $\sum_{n\geq 0} u_n$ *by grouping the terms.*

We're interested here in the possible links between the natures of $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ and, in the case of convergence, the links of their sums. We then have

Proposition 1.8.2 *If the series* $\sum_{n\geq 0} u_n$ *converges then* $\sum_{n\geq 0} v_n$ *converges, and furthermore we have,*

$$\sum_{n=0}^{+\infty} u_n = \sum_{n=0}^{+\infty} v_n$$

Proof: Let $(S_n)_n$ and $(T_n)_n$ be the sequences of partial sums of $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ respectively i.e.,

$$S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + ... + u_n,$$

et

$$T_n = \sum_{k=0}^{n} v_k = v_0 + u_1 + ... + v_n.$$

Thus,

$$T_0 = v_0 = u_0 + u_1 + ... + u_{n_1} = S_{n_1},$$
$$T_1 = v_0 + v_1 = u_0 + u_1 + ... + u_{n_1} + u_{n_1+1} + ... + u_{n_2} = S_{n_2},$$
$$\vdots$$
$$T_k = v_0 + v_1 + ... + v_k = u_0 + ... + u_{n_{k+1}} = S_{n_{k+1}}.$$

Where the sequence $(T_n)_n$ is a sub-sequence of $(S_n)_n$. Since the series $\sum_{n\geq 0} u_n$ converges i.e.,

$$\exists S \in \mathbb{R}/ \lim_{n\to+\infty} S_n = S.$$

Then the subset $(T_n)_n$ of $(S_n)_n$ also converges, and we have

$$\lim_{n\to+\infty} T_n = S.$$

Where

$$\sum_{n=0}^{+\infty} v_n = S.$$

□

Remark 1.8.3 *The reciprocal of the previous proposition is false, i.e.,*

$$\sum_{n\geq 0} v_n \text{ converge } \nRightarrow \sum_{n\geq 0} u_n \text{ converge},$$

as shown in the following example:

Example 1.8.4 *The series* $\sum_{n\geq 0} u_n = \sum_{n\geq 0} (-1)^n$ *is divergent because its sequence of partial sums* $(S_n)_n$ *defined by*

$$S_n = \begin{cases} 1 \text{ if} n \text{ is even} \\ 0 \text{ if} n \text{ is odd} \end{cases}$$

has no limit. But the $\sum_{n\geq 0} v_n$ *series obtained by grouping terms defined by*

$$\sum_{n\geq 0} v_n = (1-1) + (1-1) + ... + (1-1) + ...$$

converges since it's the null series.

Remark 1.8.5 *The preceding proposition is of little interest, since it assumes convergence of the series* $\sum_{n\geq 0} u_n$.

This gives us the term grouping theorem.

Theorem 1.8.6 *Let* $\sum_{n\geq 0} u_n$ *be a infinite series and* $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ *a strictly increasing application; note for* $n \in \mathbb{N}$,

$$v_n = \sum_{p=\varphi(n)}^{\varphi(n+1)-1} u_p.$$

If $u_n \underset{n\to\infty}{\rightarrow} 0$ *and* $(\varphi(n+1) - \varphi(n))_n$ *is bounded then the series* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *are of the same nature, and in the case of convergence, we have*

$$\sum_{n=0}^{+\infty} v_n = \sum_{p=\varphi(0)}^{+\infty} u_p.$$

Proof: It is assumed that $\sum_{ngeq0} v_n$ is convergent. Let $N \in \mathbb{N}$ be such that $N \geq \varphi(0)$. Since $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly increasing application, there exists a unique $n_N \in \mathbb{N}$ such that

$$\varphi(n_N) \leq N < \varphi(n_N + 1).$$

Then,

$$\sum_{p=\varphi(0)}^{N} u_p = \sum_{p=\varphi(0)}^{\varphi(n_N)-1} u_p + \sum_{p=\varphi(n_N)}^{N} u_p. \tag{1.17}$$

Now for all $N \in \mathbb{N}$, we have

$$\sum_{n=0}^{N} v_n = \sum_{n=0}^{N} \left(\sum_{p=\varphi(n)}^{\varphi(n+1)-1} u_p \right) = \sum_{p=\varphi(0)}^{\varphi(N+1)-1} u_p. \tag{1.18}$$

From (1.17) into (1.18), we deduce that

$$\sum_{p=\varphi(0)}^{N} u_p = \sum_{n=0}^{n_N-1} v_n + \sum_{p=\varphi(n_N)}^{N} u_p.$$

Clearly

$$\sum_{n=0}^{n_N-1} v_n \underset{N\to\infty}{\to} \sum_{n=0}^{+\infty} v_n,$$

since $\sum_{n\geq 0} v_n$ converges. To conclude, all we need to know is that

$$\sum_{p=\varphi(n_N)}^{N} v_p \underset{N\to\infty}{\to} 0. \tag{1.19}$$

For this, we have

$$\left| \sum_{p=\varphi(n_N)}^{N} u_p \right| \leq \sum_{p=\varphi(n_N)}^{N} |u_p| \leq \sum_{p=\varphi(n_N)}^{\varphi(n_N+1)-1} |u_p|. \tag{1.20}$$

Thus, by passing to the limit in (1.20) when $N \to +\infty$, we conclude (1.19) because $u_p \underset{p\to\infty}{\to} 0$ and $(\varphi(n+1) - \varphi(n))_n$ is bounded.

□

1.9 Permutation of Terms

We're now interested in the following question: is it possible, without changing the nature or the sum of a series, to permute the order of the terms in the series? We can use examples to show that manipulating the terms of a series in this way can change the nature and sum of the series unchanged. Consider the series $\sum_{n\geq 0} u_n$, with $u_n \in \mathbb{N}$, for all $n \in \mathbb{N}$. By permuting the terms (without repetition), we obtain a series of the form :

$$\sum_{n\geq 0} v_n \text{ where } v_n = u_{\sigma(n)},$$

and $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection (also called a permutation of $\mathbb{N}$). The *series* $\sum_{n\geq 0} v_n$ is deduced from the *series* $\sum_{n\geq 0} u_n$ by changing the order of the terms.

Theorem 1.9.1 *Let* $\sum_{k=0}^{+\infty} u_k$ *be an absolutely convergent series and let* S *be its sum. Let* $\sigma : \mathbb{N} \to \mathbb{N}$ *be a bijection of the set of indices. Then the series* $\sum_{k=0}^{+\infty} u_{\sigma(k)}$ *converges and*

$$\sum_{k=0}^{+\infty} u_{\sigma(k)} = S$$

Remark: the condition of absolute convergence is indispensable. As it happens, for a series that is convergent, but not absolutely convergent, we can permute the terms to obtain any value! As an example of permutation, we can reorder the terms $u_0, u_1, u_2, u_3, \ldots$ by taking two even-ranked terms and then one odd-ranked term, which gives :

$$u_0, u_2, u_1, u_4, u_6, u_3, u_8, u_{10}, u_5, \ldots$$

On the other hand, it is not permitted to group all even terms first and then odd terms:

$$u_0, u_2, u_4, \ldots, u_{2k}, \ldots, u_1, u_3, \ldots, u_{2k+1}, \ldots$$

Proof: By hypothesis $\sum_{k=0}^{+\infty} |u_k|$ converges. According to Cauchy's criterion,

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \qquad \sum_{n=n_0+1}^{+\infty} |u_k| < \varepsilon.$$

Let $S = \sum_{k=0}^{+\infty} u_k$. Let $\varepsilon > 0$. Let's choose $k_0 \in \mathbb{N}$ such that $\{0, 1, 2, \ldots, n_0\} \subset \{\sigma(0), \sigma(1), \ldots, \sigma(k_0)\}$.
For $n \geq k_0$ we have :

For the first term we have

$$\left| S - \sum_{k=0}^{n} u_{\sigma(k)} \right| \leqslant \left| S - \sum_{k=0}^{n_0} u_k \right| + \left| \sum_{k=0}^{n_0} u_k - \sum_{k=0}^{n} u_{\sigma(k)} \right|$$

$$\left| S - \sum_{k=0}^{n_0} u_k \right| = \left| \sum_{k=n_0+1}^{+\infty} u_k \right| \leqslant \sum_{k=n_0+1}^{+\infty} |u_k| \leqslant \varepsilon.$$

For the second term :

$$\left| \sum_{k=0}^{n} u_{\sigma(k)} - \sum_{k=0}^{n_0} u_k \right| = \left| \sum_{k\in\{\sigma(0),\ldots,\sigma(n)\}\setminus\{0,\ldots,n_0\}} u_k \right| \leqslant \sum_{k\in\{\sigma(0),\ldots,\sigma(n)\}\setminus\{0,\ldots,n_0\}} |u_k|$$
$$\leqslant \sum_{k>n_0} |u_k| = \sum_{k=n_0+1}^{+\infty} |u_k| \leqslant \varepsilon.$$

This proves $\left| S - \sum_{k=0}^{n} u_{\sigma(k)} \right| \leqslant 2\varepsilon$ and gives the result. □

Example 1.9.2 *The aim of this example is to show that if the series is not absolutely convergent, some strange phenomena appear. appear. Remember that the alternating harmonic series converges:*

$$\sum_{k=0}^{+\infty}(-1)^k \frac{1}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Let's denote S as its sum (in fact $S = \ln 2$). If we group the terms of this series in packets of 3, and simplify, then we find half the sum!

$$\begin{aligned}
&(1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + (\frac{1}{5} - \frac{1}{10} - \frac{1}{12}) + ... + (\frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}) + ...\\
&= (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{6} - \frac{1}{8}) + (\frac{1}{10} - \frac{1}{12}) + ... + (\frac{1}{4k-2} - \frac{1}{4k}) + ...\\
&= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...)\\
&= \frac{1}{2}S.
\end{aligned}$$

1.9.1 Abel's Summation Theorem

Abel's summation theorem applies to certain series that are convergent but not absolutely convergent. It is a theorem that applies to series of the form $\sum a_k b_k$ and is stronger than Leibniz's criterion for alternating series, but it is also more difficult to implement.

Theorem 1.9.3 (Abel's summation theorem) *Let $(u_k)_{k \geqslant 0}$ and $(v_k)_{k \geqslant 0}$ be two sequences such that :*

1. *The sequence $(u_k)_{k \geqslant 0}$ is a decreasing sequence of positive reals that tends to 0.*
2. *The partial sums of the sequence $(v_k)_{k \geqslant 0}$ are bounded:*

$$\exists M \quad \forall n \in \mathbb{N} \quad |v_0 + \cdots + v_n| \leqslant M.$$

Then the series $\sum_{k \geqslant 0} u_k v_k$ converges.

Leibniz's criterion for alternating series is a special case: if $v_k = (-1)^k$ then $\left|\sum_{k=0}^{n} v_k\right| \leqslant 1$. So if (u_k) is a positive, decreasing sequence that tends to 0, then $\sum u_k v_k$ converges.

Proof: The idea of the demonstration is to make a change in the summation, akin to integration by parts. For any $n \geqslant 0$, let $V_n = v_0 + \cdots + v_n$. By hypothesis, the sequence (V_n) is bounded. We write the partial sums of the series $\sum u_k v_k$ in the following form:

$$\begin{aligned}
S_n &= u_0 v_0 + u_1 u_1 + \cdots + u_{n-1} v_{n-1} + u_n v_n\\
&= u_0 V_0 + u_1 (V_1 - V_0) + \cdots + u_{n-1} (V_{n-1} - V_{n-2}) + u_n (V_n - V_{n-1})\\
&= V_0 (u_0 - u_1) + V_1 (u_1 - u_2) + \cdots + V_{n-1} (u_{n-1} - u_n) + V_n u_n.
\end{aligned}$$

Since (V_n) is bounded, and u_n tends to 0, the last term $V_n u_n$ tends to 0. We'll show that the series $\sum V_k (u_k - u_{k+1})$ is absolutely convergent. This is because

$$|V_k (u_k - u_{k+1})| = |V_k| (u_k - u_{k+1}) \leqslant M (u_k - u_{k+1}),$$

because the sequence (u_k) is a decreasing sequence of positive reals, and $|V_k|$ is bounded by M. Now

$$M(u_0 - u_1) + \cdots + M(u_n - u_{n+1}) = M(u_0 - u_{n+1}),$$

which tends to Mu_0 since (u_k) tends to 0. The series $\sum M(u_k - u_{k+1})$ converges, so the series $\sum |V_k(u_k - u_{k+1})|$ also, by the theorem 1.4.1. So the series $\sum V_k(u_k - u_{k+1})$ is convergent, so the series (S_n) is convergent, which proves that the series $\sum u_k v_k$ converges. □

Example 1.9.4 *Study the nature of the series*

$$\sum_{n\geq 1} \frac{cosnx}{n}, \quad x \in \mathbb{R}.$$

Step 1: Define the Series

We need to determine the nature of the series:

$$S(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n}.$$

Step 2: Identify the Sequences

To apply **Abel's summation theorem**, we will identify two sequences:

- Let $u_n = \frac{1}{n}$. This sequence is positive, decreasing, and tends to 0 as $n \to \infty$.
- Let $v_n = \cos(nx)$. This sequence is oscillatory.

Step 3: Check the Conditions of Abel's Theorem

We need to verify the two conditions required by Abel's theorem:

1. **Condition on u_n**: The sequence $u_n = \frac{1}{n}$ satisfies:
   - It is positive: $u_n > 0$ for all $n \geq 1$.
   - It is decreasing: $u_n > u_{n+1}$ since $\frac{1}{n} > \frac{1}{n+1}$.
   - It tends to 0: $\lim_{n\to\infty} u_n = 0$.
2. **Condition on Partial Sums of v_n**: We need to analyze the partial sums:

$$S_N = \sum_{n=1}^{N} \cos(nx).$$

   The series of cosine terms can be evaluated using the formula for the sum of a geometric series.

Step 4: Sum of Cosine Terms

Using the relation $\cos(nx) = \text{Re}(e^{inx})$:

$$S_N = \sum_{n=1}^{N} \cos(nx) = \text{Re}\left(\sum_{n=1}^{N} e^{inx}\right).$$

The sum $\sum_{n=1}^{N} e^{inx}$ is a finite geometric series with first term e^{ix} and common ratio e^{ix}:

$$\sum_{n=1}^{N} e^{inx} = e^{ix}\frac{1-e^{iNx}}{1-e^{ix}}.$$

Step 5: Evaluating the Real Part

Taking the real part gives:

$$S_N = \text{Re}\left(\frac{e^{ix}(1-e^{iNx})}{1-e^{ix}}\right).$$

Using the fact that $e^{ix} = \cos(x) + i\sin(x)$:

$$1 - e^{ix} = 1 - \cos(x) - i\sin(x).$$

Thus, the modulus of the denominator is:

$$|1-e^{ix}| = \sqrt{(1-\cos(x))^2 + \sin^2(x)} = \sqrt{2(1-\cos(x))} = 2|\sin(x/2)|.$$

Step 6: Bound the Partial Sums

The expression for the partial sums of the series becomes:

$$|S_N| \leq \frac{1}{|1-e^{ix}|}|1-e^{iNx}| \leq \frac{1}{2|\sin(x/2)|}.$$

Since $|1-e^{iNx}|$ is bounded (oscillating terms), S_N is also bounded for all x such that $x \neq 2\pi k$ (where $k \in \mathbb{Z}$).

Step 7: Conclusion by Abel's Theorem

Since both conditions of Abel's summation theorem are satisfied:

1. $u_n = \frac{1}{n}$ is positive, decreasing, and tends to 0.
2. The partial sums $S_N = \sum_{n=1}^{N} \cos(nx)$ are bounded.

We conclude that the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$$

converges conditionally for all $x \in \mathbb{R} \setminus \{2\pi k : k \in \mathbb{Z}\}$.

Final Result

Thus, the series converges conditionally, and we can express the result as:

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} \text{ converges for all } x \in \mathbb{R} \text{ except at points } x = 2\pi k, k \in \mathbb{Z}.$$

1.10 Exercises of the Chapter

Exercise 1.10.1 *Study of the geometric series* $\sum_{n=1}^{\infty} aq^{n-1}$, *where* $a, q \in \mathbb{R}$.

Correction 1.10.1 *Note that* $S_n = a + aq + \ldots + aq^{n-1}$ *for* $n \in \mathbb{N}$. *Clearly, if* $a = 0$, *then* $S_n = 0$ *for all* $n \in \mathbb{N}$. *Hence, assume that* $a \neq 0$. *Then we have*

$$S_n = \begin{cases} na & \text{if} \quad q = 1 \\ \dfrac{a(1-q^n)}{1-q} & \text{if} \quad q \neq 1 \end{cases}$$

Thus, if $q = 1$, *then* (S_n) *is not bounded; hence not convergent. If* $q = -1$, *then we have*

$$S_n = \begin{cases} a & \text{if} \quad n \text{ odd} \\ 0 & \text{if} \quad n \text{ even.} \end{cases}$$

Thus, (S_n) *diverges for* $q = -1$ *as well. Now, assume that* $|q| \neq 1$. *In this case, we have*

$$\left| S_n - \frac{a}{1-q} \right| = \frac{|a|}{|1-q|} |q|^n$$

This shows that, if $|q| < 1$, *then* (S_n) *converges to* $\frac{a}{1-q}$, *and if* $|q| > 1$, *then* (S_n) *is not bounded, hence diverges.*

Exercise 1.10.2 *Study the nature of the series with general term* u_n, *using the indicated convergence test.*

a) Necessary condition for convergence: $\lim_{n \to +\infty} u_n = 0$

$$u_n = \frac{n^3}{n^3+1}; \quad w_n = \arctan \frac{n^3+1}{n^3+2}; \quad t_n = e^{-\sqrt{n}}.$$

b) Comparison test

$$u_n = \left(\sqrt[n]{2} + \sqrt[n]{3} \right)^{-n^2}; \quad v_n = \frac{1}{\ln(n+1)}; \quad w_n = \frac{\ln(n)}{n^2}.$$

c) Equivalence test

$$u_n = \frac{n+\sqrt{n}}{2n^3-1}; \quad v_n = \sin^3 \frac{1}{n}; \quad w_n = \frac{1}{n^{1+\frac{1}{n}}}; \quad t_n = 1 - \cosh \frac{1}{\sqrt{n}}.$$

d) *Cauchy test*

$$u_n = \left(\frac{1}{2} + \frac{1}{n}\right)^n; \quad v_n = \left(\arcsin\frac{1}{n}\right)^n.$$

e) *d'Alembert test*

$$u_n = \frac{1}{(2n-1)2^{2n-1}}; \quad v_n = \frac{3^n n!}{n^n}; \quad w_n = \frac{n4}{n!}.$$

Correction 1.10.2

a) *We have* $\lim_{n\to+\infty} u_n = 1 \neq 0$, *so the series* $\sum u_n$ *diverges, and since it is a positive term series, we can write* $\sum_{n\geq 0} u_n = +\infty$, *as its partial sums are increasing and unbounded. For the series with general term* w_n, *we have*

$$\lim_{n\to+\infty} w_n = \arctan\frac{n^3+1}{n^3+2} = \frac{\pi}{4} \neq 0, \quad \textit{so} \quad \sum_{n\geq 0} w_n \textit{ diverges.}$$

For the series with general term t_n, *we have* $\lim_{n\to+\infty} e^{-\sqrt{n}} = 0$; *thus, we cannot conclude anything about the nature of the series, and a comparison test is needed.*

b) *For the comparison test, the most commonly used reference series are those of Riemann. We also recall the following frequently used results:*

$$\lim_{n\to+\infty}(n^\alpha e^{-n^\beta}) = 0 \quad \textit{for all } \alpha \in \mathbb{R} \textit{ and } \beta > 0; \quad \lim_{n\to+\infty}\frac{\ln n}{n^\alpha} = 0 \quad \textit{for all } \alpha > 0.$$

Thus, we have $\lim_{n\to+\infty} n^2 e^{-\sqrt{n}} = 0$, *implying that* $0 \leq e^{-\sqrt{n}} < n^{-2}$ *for* $n \geq N$, *so the series* $\sum e^{-\sqrt{n}}$ *from part (a) converges.*
We have $2^{\frac{1}{n}} > 1$ *and* $3^{\frac{1}{n}} > 1$, *so* $2^{\frac{1}{n}} + 3^{\frac{1}{n}} > 2$ *and* $0 < u_n < 2^{-n^2} < \frac{1}{n^2}$ *for* $n \geq N$, *hence* $\sum u_n$ *converges.*
We have $v_n = \frac{1}{\ln(n+1)} > \frac{1}{n}$ *for* $n \geq N$ *since* $\lim_{n\to+\infty}\frac{n}{\ln(n+1)} = +\infty$; *but* $\sum \frac{1}{n}$ *diverges, implying* $\sum v_n$ *diverges.*
As $\frac{n^\alpha \ln n}{n^2} \to 0$ *as* $n \to +\infty$ *if* $2 - \alpha > 0$, *we have* $0 \leq w_n < n^{-\alpha}$ *for* $n \geq N$ *and any* α, $0 < \alpha < 2$. *Taking* $1 < \alpha < 2$, *we obtain the convergence of the series* $\sum w_n$ *by comparison with the convergent Riemann series* $\sum n^{-\alpha}$.

c) *The equivalence test is a special case of the previous one: it mainly applies when the general term of the series* u_n *has an asymptotic equivalent, for* $n \to +\infty$, *in terms of powers:* $n^{-\alpha}$. *Here, for* $n \to +\infty$,

$$u_n \sim \frac{1}{2n^2} \quad \Rightarrow \quad \sum u_n \textit{ converges.} v_n \sim \frac{1}{n^3} \quad \Rightarrow \quad \sum v_n \textit{ converges.} w_n \sim \frac{1}{n},$$

$$\textit{since} \lim_{n\to+\infty} \sqrt[n]{n} = 1 \quad \Rightarrow \quad \sum w_n \textit{ diverges.} t_n \sim \frac{1}{2n} \quad \Rightarrow \quad \sum t_n \textit{ diverges.}$$

d) *We have* $\sqrt[n]{u_n} = \left(\frac{1}{2} + \frac{1}{n}\right) \to \frac{1}{2} < 1$ *as* $n \to +\infty$, *so* $\sum u_n$ *converges.*
We have $\sqrt[n]{v_n} = \arcsin(n^{-1}) \to 0$ *as* $n \to +\infty$, *so* $\sum v_n$ *converges.*
Note: This test mainly applies to expressions with n-th powers.

e) For $u_n = \dfrac{1}{(2n-1)2^{2n-1}}$, *we have*

$$\frac{u_{n+1}}{u_n} = \frac{2n-1}{4(2n+1)} \to \frac{1}{4} \quad \text{as } n \to +\infty.$$

Since $\frac{1}{4} < 1$, $\sum u_n$ *converges.*

For $v_n = v_n = \dfrac{3^n n!}{n^n}$, *we have*

$$\frac{v_{n+1}}{v_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} . \frac{n^n}{3^n n!} = \frac{3}{(1+\frac{1}{n})^n} \to \frac{3}{e} < 1 \quad \text{as } n \to +\infty,$$

so $\sum v_n$ *converges.*

For $w_n = \dfrac{n^3}{n!}$, *we have*

$$\frac{w_{n+1}}{w_n} = \left(\frac{n+1}{n}\right)^4 \cdot \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to +\infty : \quad \Rightarrow \sum w_n \text{ converges.}$$

The d'Alembert ratio test is particularly useful for expressions involving factorials or powers.

Exercise 1.10.3 *To study the nature of the series whose general term is given by*

$$u_n = \frac{1! + 2! + \cdots + n!}{(n+p)!},$$

Correction 1.10.3

- *For* $p = 0$*:*

$$u_n = \frac{1! + 2! + \cdots + n!}{n!} = 1 + \frac{1! + 2! + \cdots + (n-1)!}{n!} > 1$$

u_n *does not tend to* 0*, so* $\sum u_n$ *diverges grossly for* $p = 0$.

- *For* $p = 1$*:*

$$u_n = \frac{1}{(n+1)!} + \frac{2!}{(n+1)!} + \cdots + \frac{(n-1)!}{(n+1)!} + \frac{n!}{(n+1)!}$$

$$u_n \geq \frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Since $\sum \frac{1}{n+1}$ *diverges,* $\sum u_n$ *also diverges for* $p = 1$.

- *For* $p = 2$*:*

$$u_n = \frac{1}{(n+2)!} + \frac{2!}{(n+2)!} + \cdots + \frac{(n-1)!}{(n+2)!} + \frac{n!}{(n+2)!}$$

One might be tempted to say that we have a sum of convergent series, so $\sum u_n$ *converges. Unfortunately, the number of terms grows with* n*, leading to an infinite number of terms, so we cannot conclude anything.*

$$u_n = \sum_{k=1}^{n} \frac{k!}{(n+2)!} = \sum_{k=1}^{n-1} \frac{k!}{(n+2)!} + \frac{n!}{(n+2)!} \leq \frac{n(n-1)!}{(n+2)!} + \frac{n!}{(n+2)!}$$

$$u_n \leq 2\frac{n!}{(n+2)!} = \frac{2}{(n+1)(n+2)} \sim \frac{2}{n^2}$$

Since $\sum \frac{1}{n^2}$ *converges, we have* $\sum u_n$ *converging for* $p = 2$.

- *For* $p \geq 3$*:*
$$u_n = \frac{1! + 2! + \cdots + n!}{(n+p)!} \leq \frac{n\, n!}{(n+p)!} = \frac{n\, n!}{n!(n+1)\cdots(n+p)}$$
By simplifying with $n!$ *and letting* $u_n \leq \dfrac{n}{(n+1)\cdots(n+p)}$, *we have*
$$\frac{n}{(n+1)\cdots(n+p)} \sim \frac{n}{n^p} = \frac{1}{n^{p-1}} \text{ for } p \geq 3$$
Since $\sum \frac{1}{n^{p-1}}$ *is a convergent Riemann series because* $p-1 \geq 2$, *thus* $\sum u_n$ *converges for* $p \geq 3$.
Note: One can also observe that u_n *(when* $p \geq 3$*) is bounded above by* u_n *(when* $p = 2$*), and the latter is convergent.*

Exercise 1.10.4 *Calculate the sums of the following series, showing their convergence:*

1. $\sum_{n \geq 0} (n+1)3^{-n}$
2. $\sum_{n \geq 0} \frac{n}{n^4 + n^2 + 1}$
3. $\sum_{n \geq 3} \frac{2n-1}{n^3 - 4n}$

Correction 1.10.4

1. *Let* $S_n = \sum_{k=0}^{n} (k+1)3^{-k}$. *The idea is to calculate the sum of* $(1-3^{-1})S_n$. *We have:*
$$\begin{aligned}(1-3^{-1})S_n &= (1-3^{-1}) \sum_{k=0}^{n} (k+1)3^{-k} \\ &= \sum_{k=0}^{n} (k+1)3^{-k} - \sum_{k=0}^{n} (k+1)3^{-(k+1)} \\ &= \sum_{k=0}^{n} k3^{-k} + \sum_{k=0}^{n} 3^{-k} - \sum_{k=0}^{n} (k+1)3^{-(k+1)}\end{aligned}$$
By re indexing the sums, we obtain:
$$\begin{aligned}(1-3^{-1})S_n &= \sum_{k=1}^{n} k3^{-k} + \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k - \sum_{k=1}^{n} k3^{-k} - (n+1)3^{-(n+1)} \\ &= \sum_{k=0}^{n} \frac{1}{3^k} - \frac{n+1}{3^{n+1}} = \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{\frac{2}{3}} - \underbrace{\frac{n+1}{3^{n+1}}}_{\to 0}\end{aligned}$$
as the sum of the terms of a geometric series with ratio $\frac{1}{3} \in]-1, 1[$. *Therefore,*
$$\lim_{n \to +\infty} \left(1 - \frac{1}{3}\right) S_n = \frac{3}{2}$$
which gives
$$\sum_{k=0}^{+\infty} (k+1)3^{-k} = \frac{9}{4} \cdot$$
Note: We recognize the first derivative of the geometric series with ratio $\frac{1}{3}$.

2. *Let* $u_n = \dfrac{n}{n^4+n^2+1}$ *and seek to decompose it into partial fractions.*

$$\begin{aligned} n^4+n^2+1 &= (n^4+2n^2+1)-n^2 = (n^2+1)^2-n^2 \\ &= (n^2+n+1)(n^2-n+1) \end{aligned}$$

Thus, $u_n = \dfrac{n}{(n^2+n+1)(n^2-n+1)}$.
Let's find A *and* $B \in \mathbb{R}$ *such that* $u_n = \dfrac{A}{n^2+n+1} + \dfrac{B}{n^2-n+1}$, *which leads to* $A(n^2-n+1)+B(n^2+n+1) = n$. *This is equivalent to* $(A+B)n^2+(B-A)n+(A+B) = n$. *By identification, we have:*

$$\begin{cases} A+B=0 \\ \\ B-A=1 \\ \\ A+B=0 \end{cases} \iff \begin{cases} A = -\frac{1}{2} \\ \\ \\ B = \frac{1}{2} \end{cases}$$

Thus:

$$u_n = \frac{n}{n^4+n^2+1} = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)$$

$$\sum_{n=0}^{N} u_n = \sum_{n=0}^{N} \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\left(\sum_{n=0}^{N}\frac{1}{n^2-n+1} - \sum_{n=0}^{N}\frac{1}{n^2+n+1}\right)$$

Now, $n^2+n+1 = (n+1)^2-(n+1)+1$. *By re indexing the second sum:*

$$\begin{aligned} \sum_{n=0}^{N} u_n &= \frac{1}{2}\left(\sum_{n=0}^{N}\frac{1}{n^2-n+1} - \sum_{n=1}^{N+1}\frac{1}{n^2-n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{(N+1)^2-(N+1)+1}\right) \quad \textit{by telescoping.} \end{aligned}$$

The series is therefore convergent, and the sum is

$$\sum_{n=0}^{+\infty} \frac{n}{n^4+n^2+1} = \frac{1}{2}\,.$$

3. *Let's decompose* $v_n = \dfrac{2n-1}{n^3-4n}$ *into partial fractions. Since*

$$n^3-4n = n(n^2-4) = n(n-2)(n+2),$$

let's find α, β, *and* $\gamma \in \mathbb{R}$ *such that:*

$$v_n = \frac{\alpha}{n} + \frac{\beta}{n+2} + \frac{\gamma}{n-2}.$$

So,

$$\begin{aligned} 2n-1 &= \alpha(n-2)(n+2) + \beta n(n-2) + \gamma n(n+2) \\ &= (\alpha+\beta+\gamma)n^2 + (2\gamma-2\beta)n - 4\alpha \end{aligned}$$

By identification:

$$\begin{cases} \alpha+\beta+\gamma=0 \\ 2(\gamma-\beta)=2 \\ -4\alpha=-1 \end{cases} \Longleftrightarrow \begin{cases} \alpha &= \frac{1}{4} \\ \gamma &= 1+\beta \\ \alpha+2\beta+1 &= 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha &= \frac{1}{4} \\ \beta &= -\frac{5}{8} \\ \gamma &= \frac{3}{8} \end{cases}$$

Thus

$$\frac{2n-1}{n^3-4n} = \frac{1}{4n} - \frac{5}{8(n+2)} + \frac{3}{8(n-2)}$$

and

$$\begin{aligned} \sum_{n=3}^{N} v_n &= \sum_{n=3}^{N} \left(\frac{1}{4n} - \frac{5}{8(n+2)} + \frac{3}{8(n-2)} \right) \\ &= \sum_{n=3}^{N} \frac{1}{4n} - \sum_{n=3}^{N} \frac{5}{8(n+2)} + \sum_{n=3}^{N} \frac{3}{8(n-2)}. \end{aligned}$$

Let's re index the last two sums:

$$\sum_{n=3}^{N} v_n = \frac{1}{8}\left[2\sum_{n=3}^{N} \frac{1}{n} - 5\sum_{n=5}^{N+2} \frac{1}{n} + 3\sum_{n=1}^{N-2} \frac{1}{n} \right]$$

Then, by telescoping,

$$\sum_{n=3}^{N} v_n = \frac{1}{8}\left(\frac{89}{12} - \frac{3}{N-1} - \frac{3}{N} - \frac{5}{N+1} - \frac{5}{N+2} \right) \xrightarrow[n\to+\infty]{} \frac{89}{12}.$$

Thus, the series converges and $\displaystyle\sum_{n=3}^{+\infty} \frac{2n-1}{n^3-4n} = \frac{89}{96}$.

Exercise 1.10.5 *Finding an asymptotic equivalent of* $S_n = \displaystyle\sum_{1}^{n} f(k)$ *by comparison with an integral*

a) Let f *be a continuous, positive, and increasing function on* $[0,+\infty[$.
Show that for all $n \geq 1$, *we have:*

$$\int_0^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq \int_1^{n+1} f(x)\,dx$$

b) Find an asymptotic equivalent for $n \to +\infty$ *of*

$$S_n = 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}.$$

Deduce the nature of the series with general term $u_n = \dfrac{1}{S_n}$.

Correction 1.10.5

a) *For all $k = 0, 1, \dots, n$ and for any $x \in [k, k+1]$, we have $f(k) \le f(x) \le f(k+1)$, from which follows*

$$f(k) \le \int_k^{k+1} f(x)\,dx \le f(k+1).$$

Therefore,

$$\int_0^n f(x)\,dx = \sum_{k=0}^{n-1} \int_k^{k+1} f(x)\,dx \le \sum_{k=0}^{n-1} f(k+1) = \sum_{k=1}^{n} f(k)$$

and

$$\sum_{k=1}^{n} f(k) \le \sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx = \int_1^{n+1} f(x)\,dx.$$

b) *Let us consider the function $f(x) = \sqrt{x}$. Using part (a), we get*

$$\frac{2}{3} n^{\frac{3}{2}} = \int_0^n \sqrt{x}\,dx \le S_n \le \int_1^{n+1} \sqrt{x}\,dx = \frac{2}{3}[(n+1)^{\frac{3}{2}} - 1].$$

From this, we have $1 \le \left(\frac{2}{3} n^{\frac{3}{2}}\right)^{-1} S_n \le n^{-\frac{3}{2}} \left[(n+1)^{\frac{3}{2}} - 1\right] \to 1$ as $n \to +\infty$. This shows that

$$S_n \sim \frac{2}{3} n^{\frac{3}{2}} \text{ as } n \to +\infty.$$

The series $\sum u_n$ has positive terms, and $u_n \sim \frac{3}{2n^{\frac{3}{2}}}$, so

$$\sum \frac{1}{n^{\frac{3}{2}}} \text{ converges } \Rightarrow \sum u_n \text{ converges.}$$

c) *Let $f(x) = x^\alpha$ $(\alpha > 0)$; f is continuous and increasing on $[0, +\infty]$. We have*

$$\int_0^n f(x)\,dx \le \sum_{k=1}^{n} f(k) \le \int_1^{n+1} f(x)\,dx$$

$$\frac{n^{\alpha+1}}{\alpha+1} \le S_n \le \frac{(n+1)^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1}.$$

We deduce that

$$1 \le \frac{(\alpha+1)S_n}{n^{\alpha+1}} \le \left(\frac{n+1}{n}\right)^{\alpha+1} - \frac{1}{n^{\alpha+1}},$$

from which we conclude $\lim_{n\to+\infty} \frac{(\alpha+1)S_n}{n^{\alpha+1}} = 1$, i.e., $S_n \sim \frac{n^{\alpha+1}}{\alpha+1}$ as $n \to +\infty$. For $\alpha = \frac{1}{2}$, we recover the result of part (b). Hence, $\lim_{n\to+\infty} \frac{S_n}{n^{\alpha+1}} = \frac{1}{\alpha+1}$. Let $n \ge 1$, then $\frac{S_n}{n^{\alpha+1}} = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^\alpha \to \frac{1}{n}$ as $\alpha \to +\infty$.

Exercise 1.10.6 *Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.*

Correction 1.10.6 *The Integral Test states that if $f(x)$ is positive, continuous, and decreasing for $x \geq 1$, then the convergence of the improper integral*

$$\int_1^\infty f(x)\,dx$$

determines the convergence of the series $\sum f(n)$.

Here, $f(x) = \dfrac{1}{x(\ln x)^2}$. *We evaluate the integral:*

$$\int_1^\infty \frac{1}{x(\ln x)^2}\,dx$$

Let $u = \ln x$, *hence* $du = \frac{1}{x}dx$, *and the integral becomes:*

$$\int_1^\infty \frac{du}{u^2}$$

The integral $\displaystyle\int_1^\infty u^{-2}du$ *converges, so by the Integral Test, the series converges.*

Exercise 1.10.7

a) Determine the nature of the following alternating series:

$$u_n = (-1)^n \frac{\arctan n}{1+n^2}; \quad v_n = (-1)^n \frac{n^2}{2^n}; \quad w_n = \sin\left(\left(\frac{1}{n} + n\right)\pi\right).$$

b) Study the series with general term

$$u_n = \sin\left(\pi\sqrt{n^2+1}\right).$$

Correction 1.10.7

a) We have $|u_n| \sim \dfrac{\pi}{2n^2}$ *as* $n \to +\infty$: $\sum u_n$ *converges absolutely.*
$|v_n| = \dfrac{n^2}{2^n}$. *Now,* $\dfrac{|v_{n+1}|}{|v_n|} = \left(\dfrac{n+1}{n}\right)^2 \cdot \dfrac{1}{2} \to \dfrac{1}{2}$ *as* $n \to +\infty$: $\sum v_n$ *converges absolutely by applying the d'Alembert ratio test.*
$w_n = \sin\left(\dfrac{\pi}{n}\right) \cdot \cos(n\pi) = (-1)^n \sin\left(\dfrac{\pi}{n}\right)$; *and* $|w_n| \sim \dfrac{\pi}{n}$ *as* $n \to +\infty$, *so* $\sum w_n$ *does not converge absolutely. Let's then apply the convergence test: the sequence* (a_n) *with* $a_n = \sin\left(\frac{\pi}{n}\right)$ *tends to zero and is decreasing, so* $\sum w_n$ *converges, meaning there is conditional convergence.*

b) We have $\pi\sqrt{n^2+1} = n\pi + \pi(\sqrt{n^2+1} - n) = n\pi + \dfrac{\pi}{\sqrt{n^2+1}+n}$, *so* $u_n = (-1)^n \sin\left(\dfrac{\pi}{\sqrt{n^2+1}+n}\right)$.
Now, $0 < \dfrac{\pi}{\sqrt{n^2+1}+n} < \dfrac{\pi}{2}$ *for* $n \geq 1$, *so* $\sum u_n$ *is an alternating series. We have* $\lim\limits_{n\to+\infty} u_n = 0$ *and* $|u_{n+1}| < |u_n|$ *since* $\dfrac{\pi}{\sqrt{n^2+1}+n}$ *is a decreasing function of n and the sine is an increasing function on* $[0, \dfrac{\pi}{2}]$. *Therefore,* $\sum u_n$ *converges, but not absolutely, since* $|u_n| \sim \dfrac{\pi}{2n}$ *as* $n \to +\infty$.

Exercise 1.10.8 *Let the series with general term*

$$u_n = \sqrt{1 + \frac{(-1)^n}{\sqrt{n}}} - 1, \quad n \geq 1.$$

a) *Show that* $\sum u_n$ *is an alternating series.*

b) *Using asymptotic expansions, show that* $\sum u_n$ *diverges.*

Correction 1.10.8

a) *Pour* $n = 2k$, $u_{2k} = \sqrt{1 + \frac{1}{\sqrt{2k}}} - 1 > 0.$

Pour $n = 2k+1$, $u_{2k+1} = \sqrt{1 - \frac{1}{\sqrt{2k+1}}} - 1 < 0.$

b) *Pour* $n \to +\infty$, $u_n = \left(1 + \frac{(-1)^n}{\sqrt{n}}\right)^{\frac{1}{2}} - 1 = \frac{(-1)^n}{2\sqrt{n}} - \frac{1}{8n} + o\left(\frac{1}{n^{\frac{3}{2}}}\right)$. *Or,* $\frac{(-1)^n}{2\sqrt{n}}$ *représente le terme général d'une série alternée convergente;* $\frac{-1}{8n}$ *est le terme général d'une série divergente, et le reste* $O\left(\frac{1}{n^{\frac{3}{2}}}\right)$ *est une série absolument convergente. La série* $\sum u_n$ *est donc divergente.*

Exercise 1.10.9 *Show that the series* $\sum_{n=1}^{\infty}(-2)^n \frac{1}{n}$ *converges conditionally.*

Correction 1.10.9 *We know from the Alternating Series Test that the series* $\sum_{n=1}^{\infty}(-2)^n \frac{1}{n}$ *converges.*

However, the corresponding series $\sum_{n=1}^{\infty} \frac{1}{n}$ *(the harmonic series) diverges.*
Since the series converges but not absolutely, it converges conditionally.

Exercise 1.10.10

1. *Which series are Leibniz series? Which series are convergent?*

   a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

   b) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[4]{4}} + ...$

2. *Are the following series convergent? Are the following series absolute convergent?*

   a) $\sum_{n=1}^{+\infty}(-1)^n \frac{1}{2n+1}$.

   b) $\sum_{n=1}^{+\infty} \frac{\sin n}{n^2}$.

Correction 1.10.10

1. a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ *is a Leibniz series because* $\frac{1}{n}$ *converges to 0 monotonically decreasly, therefore the series is a convergent (but not absolute convergent).*

   b) *The terms of* $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[4]{4}} + \cdots = \sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt[n]{n}}$ *are alternating, but the sequence of the terms does not converge to* 0 $(|a_n| \to 1)$*, therefore the series is divergent.*

2. a) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$ *is convergent Leibniz series, but not absolute convergent.*

b) $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ *is absolute convergent, because* $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$.

Exercise 1.10.11 *Nature of the series with general term*

a)

$$u_n = \frac{\cos n}{n^2};\ v_n = \frac{\cos n}{n};\ w_n = \frac{\cos^2 n}{n}.$$

b)

$$u_n = \cos(an)\frac{\ln(n)}{\sqrt{n}} \text{ with } a \in \mathbb{R}.$$

Correction 1.10.11

a) *We have* $|u_n| \leq \frac{1}{n^2}$, *so* $\sum |u_n|$ *converges absolutely.*
For the series $\frac{\cos n}{n}$, *we will apply Abel's theorem.*
Let $b_n = \frac{1}{n}$ *and* $A_n = \sum_{k=1}^{n}\cos k$. *Recall that* $|\sum_{k=0}^{n}\cos k\theta| \leq \frac{1}{\sin\frac{\theta}{2}}$ *for* $\theta \not\equiv [2\pi]$. *The hypotheses of Abel's theorem are verified, so* $\sum v_n$ *converges.*
We have $w_n = \frac{\cos^2 n}{n} = \frac{1+\cos 2n}{2n} = \left(\frac{1}{n} + \frac{\cos 2n}{n}\right)\cdot\frac{1}{2}$. *Now, since* $\sum \frac{1}{n}$ *diverges and* $\sum \frac{\cos 2n}{n}$ *converges by Abel's theorem, it follows that* $\sum w_n$ *diverges.*

b) *If* $a = 2k\pi$, *then* $\cos an = 1$, *so* $\frac{\ln(n)}{\sqrt{n}} > \frac{1}{\sqrt{n}}$, *for* $n \geq N$, *hence* $\frac{1}{\sqrt{n}}$ *diverges* $\Rightarrow \sum u_n$ *diverges.*
If $a \neq 2k\pi$, *let* $a_n = \cos an$ *and* $b_n = \frac{\ln n}{\sqrt{n}}$. *We have* $\lim_{n\to+\infty} b_n = 0$ *and for* $n \geq 1$, $\left(\frac{\ln n}{\sqrt{n}}\right)_n$ *is a positive decreasing sequence (note that* $\left(\frac{\ln x}{\sqrt{x}}\right)' = \frac{1}{x\sqrt{x}}\left(1 - \frac{\ln x}{2}\right) < 0$ *if* $x > e^2$*).*
We also have $|\sum_{k=0}^{n}\cos ka| \leq \frac{1}{\sin\frac{a}{2}}$; *hence, by applying Abel's theorem,* $\sum a_n b_n$ *converges.*

Chapter 2

Sequences and Series of Functions

In this chapter, we introduce and analyze the concept of convergence for sequences and series of functions. There are various ways to define the convergence of a sequence of functions, with each definition leading to distinct types of convergence. Here, we focus on two fundamental types: pointwise convergence and uniform convergence. All functions considered here are defined on an interval A of $\mathbb{R}$ and take values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
Let $\mathcal{F}(A, \mathbb{R})$ the space of functions $f : A \longrightarrow \mathbb{R}$ defined on $A \subset \mathbb{R}$.

Definition 2.0.1 *A.sequence of functions defined on A is any mapping from $\mathbb{N}$ to $\mathcal{F}(A, \mathbb{R})$ and is denoted by $(f_n)_n$.*

Example 2.0.2 *Let $A = [0, 1]$ and $f_n(x) = x^n$ $\forall n \in \mathbb{N}$. $(f_n)_n$ is a sequence of functions defined on $[0, 1]$.*

Remark 2.0.3 *For any $x \in A$, the sequence $(f_n(x))_n$ is a numerical sequence which may be convergent or divergent.*

2.0.1 Pointwise Convergence

Pointwise convergence defines the convergence of functions in terms of the convergence of their values at each point of their domain.

Definition 2.0.4 *Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ pointwise on A if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in A$.*

We say that the sequence (f_n) converges pointwise if it converges pointwise to some function f, in which case

$$f(x) = \lim_{n\to\infty} f_n(x)$$

f is called the pointwise limit of the sequence (f_n).

Pointwise convergence is, perhaps, the most obvious way to define the convergence of functions, and it is one of the most important. Nevertheless, as the following examples illustrate, it is not as well-behaved as one might initially expect.

Example 2.0.5 *Suppose that $f_n : (0, 1) \to \mathbb{R}$ is defined by*

$$f_n(x) = \frac{n}{nx + 1}$$

Then, since $x \neq 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x + 1/n} = \frac{1}{x}$$

so $f_n \to f$ *pointwise where* $f : (0,1) \to \mathbb{R}$ *is given by*

$$f(x) = \frac{1}{x}$$

We have $|f_n(x)| < n$ *for all* $x \in (0,1)$*, so each* f_n *is bounded on* $(0,1)$*, but the pointwise limit* f *is not. Thus, pointwise convergence does not, in general, preserve boundedness.*

Example 2.0.6 *Suppose that* $f_n : [0,1] \to \mathbb{R}$ *is defined by* $f_n(x) = x^n$*. If* $0 \le x < 1$*, then* $x^n \to 0$ *as* $n \to \infty$*, while if* $x = 1$*, then* $x^n \to 1$ *as* $n \to \infty$*. So* $f_n \to f$ *pointwise where*

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Although each f_n *is continuous on* $[0,1]$*, the pointwise limit* f *is not (it is discontinuous at 1). Thus, pointwise convergence does not, in general, preserve continuity.*

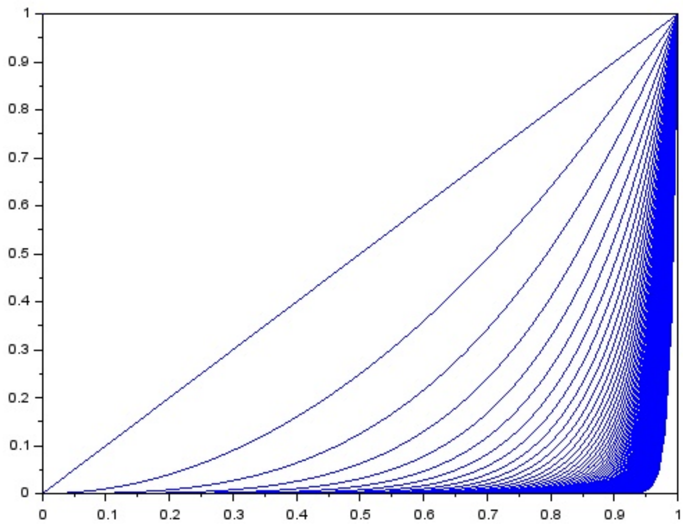


Figure 2.1: The Sequence functions x^n.

Example 2.0.7 *Define* $f_n : \mathbb{R} \to \mathbb{R}$ *by*

$$f_n(x) = \frac{\sin nx}{n}$$

Then $f_n \to 0$ *pointwise on* $\mathbb{R}$*. The sequence* (f_n') *of derivatives* $f_n'(x) = \cos nx$ *does not converge pointwise on* $\mathbb{R}$*; for example,*

$$f_n'(\pi) = (-1)^n$$

does not converge as $n \to \infty$*. Thus, in general, one cannot differentiate a pointwise convergent sequence. This behavior isn't limited to pointwise convergent sequences, and happens because the derivative of a small, rapidly oscillating function can be large.*

Example 2.0.8 *Define* $f_n : \mathbb{R} \to \mathbb{R}$ *by*

$$f_n(x) = \frac{x^2}{\sqrt{x^2 + 1/n}}$$

If $x \neq 0$, then

$$\lim_{n\to\infty} \frac{x^2}{\sqrt{x^2+1/n}} = \frac{x^2}{|x|} = |x|$$

while $f_n(0) = 0$ for all $n \in \mathbb{N}$, so $f_n \to |x|$ pointwise on $\mathbb{R}$. Moreover,

$$f_n'(x) = \frac{x^3 + 2x/n}{(x^2+1/n)^{3/2}} \to \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The pointwise limit $|x|$ isn't differentiable at 0 even though all of the f_n are differentiable on $\mathbb{R}$ and the derivatives f_n' converge pointwise on $\mathbb{R}$. (The f_n 's "round off" the corner in the absolute value function.)

Example 2.0.9 *Define $f_n : \mathbb{R} \to \mathbb{R}$ by*

$$f_n(x) = \left(1 + \frac{x}{n}\right)^n$$

Then, by the limit formula for the exponential, $f_n \to e^x$ pointwise on $\mathbb{R}$.

2.0.2 Uniform Convergence

In this section, we introduce a stronger notion of convergence of functions than pointwise convergence, called uniform convergence. The difference between pointwise convergence and uniform convergence is analogous to the difference between continuity and uniform continuity.
The pointwise convergence on I of a sequence of functions (f_n) to f is written

$$\forall x \in I,\ f(x) = \lim_{n\to\infty} f_n(x).$$

Or, by rewriting the definition of the limit:

$$\forall x \in I,\ \forall \varepsilon > 0,\ \exists \underbrace{n_0}_{\substack{\text{depends on}\\ \varepsilon \text{ and } x}} \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0 \Longrightarrow |f_n(x) - f(x)| < \varepsilon$$

While it's normal for n_0 to depend on ε (the more precise the approximation we want, the more terms of the sequence we need to calculate), it's sometimes inconvenient that it also depends on x (the sequence does not converge everywhere to f at the same rate).
This led to the following definition:

Definition 2.0.10 (Uniform Convergence) *Suppose that (f_n) is a sequence of functions $f_n : A \to \mathbb{R}$ and $f : A \to \mathbb{R}$. Then $f_n \to f$ uniformly on A if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$n > N \text{ implies that } |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in A$$

When the domain A of the functions is understood, we will often say $f_n \to f$ uniformly instead of uniformly on A.

The crucial point in this definition is that N depends only on ε and not on $x \in A$, whereas for a pointwise convergent sequence N may depend on both ε and x. A uniformly convergent sequence is always pointwise convergent (to the same limit), but the converse is not true. If a sequence converges pointwise, it may happen that for some $\varepsilon > 0$ one needs to choose arbitrarily large N 's for different points $x \in A$, meaning that the sequences of values converge arbitrarily slowly on A. In that case a pointwise convergent sequence of functions is not uniformly convergent.
The following figure gives the geometric interpretation of this definition. If the sequence (f_n) converges uniformly to f, then for n large enough, the graph of f_n stays within a tube of constant width 2ε around the graph of f:

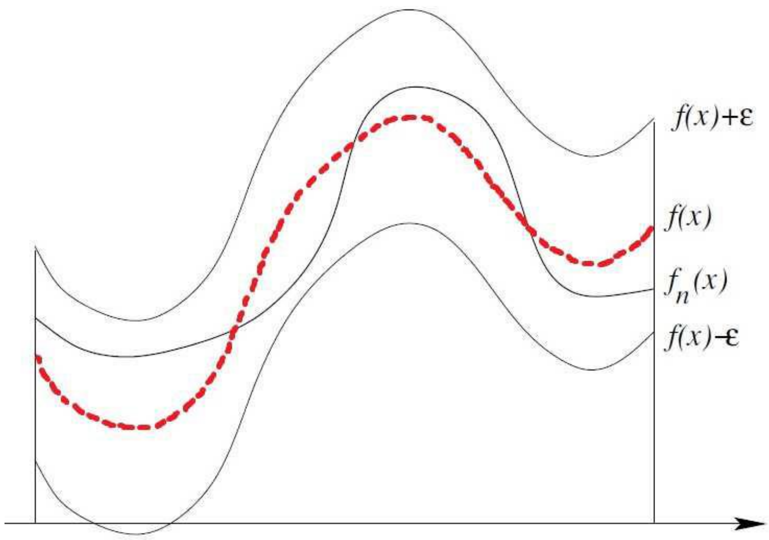


Figure 2.2: Graphical interpretation of uniform convergence

Example 2.0.11 *The sequence $f_n(x) = x^n$ in Example 2.0.6 converges pointwise on $[0, 1]$ but not uniformly on $[0, 1]$. For $0 \le x < 1$, we have*

$$|f_n(x) - f(x)| = x^n$$

If $0 < \varepsilon < 1$, we cannot make $x^n < \varepsilon$ for all $0 \le x < 1$ however large we choose n. The problem is that x^n converges to 0 at an arbitrarily slow rate for x sufficiently close to 1 . There is no difficulty in the rate of convergence at 1 itself, since $f_n(1) = 1$ for every $n \in \mathbb{N}$. As we will show, the uniform limit of continuous functions is continuous, so since the pointwise limit of the continuous functions f_n is discontinuous, the sequence cannot converge uniformly on $[0, 1]$. The sequence does, however, converge uniformly to 0 on $[0, b]$ for every $0 \le b < 1$; given $\varepsilon > 0$, we take N large enough that $b^N < \varepsilon$.

Example 2.0.12 *The functions in Example 2.0.7 converge uniformly to 0 on $\mathbb{R}$, since*

$$|f_n(x)| = \frac{|\sin nx|}{n} \le \frac{1}{n}$$

so $|f_n(x) - 0| < \varepsilon$ for all $x \in \mathbb{R}$ if $n > 1/\varepsilon$.

Example 2.0.13 *Study the uniform convergence of the following sequence of functions:*
For all $n \in \mathbb{N}^*$, we define f_n on $I = [0, +\infty[$ by

$$f_n(x) = \frac{x}{1 + n^2x^2} = \frac{1}{2n}\left(\frac{2nx}{1 + n^2x^2}\right).$$

We know that

$$(1 - nx)^2 \ge 0 \ \forall x \in \mathbb{R}, \ \forall n.$$

Then,

$$1 + n^2x^2 \ge 2nx \Rightarrow \frac{2nx}{1 + n^2x^2} \le 1.$$

Hence

$$|f_n(x) - f(x)| = \frac{1}{2n}\frac{2nx}{1 + n^2x^2} \le \frac{1}{2n},$$

with $f(x) = 0$. To make $|f_n(x) - f(x)|$ smaller than ε, it suffices to have

$$n > \frac{1}{2\varepsilon}.$$

So, it suffices to take $n_0 = \left[\frac{1}{2\varepsilon}\right] + 1$. Consequently, f_n converges uniformly to f.

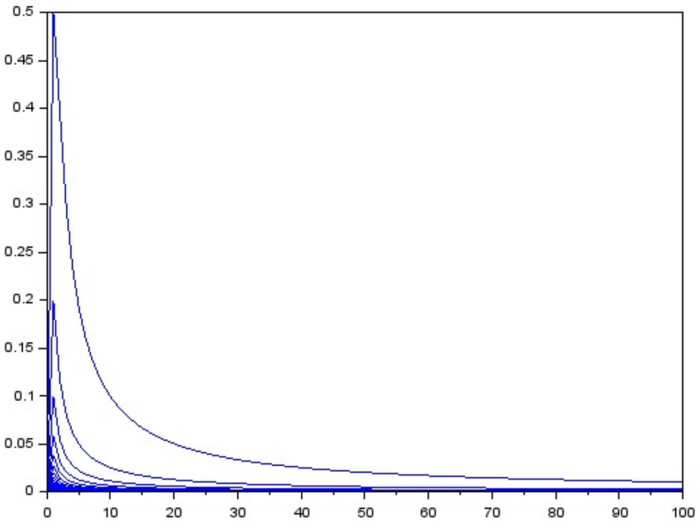


Figure 2.3: The Sequence functions $\frac{x}{1+n^2x^2}$

Proposition 2.0.14 *If (f_n) converges uniformly to f on I, then (f_n) converges pointwise to f.*

Proof: Immediate: it suffices to read the two definitions. □

Remark 2.0.15 *The converse of the proposition is false, that is, a sequence of functions can converge pointwise without converging uniformly.*

Theorem 2.0.16 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval I, taking values in $\mathbb{K}$, and f a function from I to $\mathbb{K}$.*

Then (f_n) converges uniformly to f on I if and only if

- *the functions $f_n - f$ are bounded on I (at least from a certain rank);*
- *and $\lim_{n\to+\infty} \|f_n - f\|_\infty^I = 0$, where we have set: $\|f_n - f\|_\infty^I = \sup_{x\in I} |f_n(x) - f(x)|$.*

Proof: Indeed, to say that $\|f_n - f\|_\infty^I$ exists and tends to 0 is equivalent to

$$\forall \varepsilon > 0 \ , \ \exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 \Longrightarrow \sup_{x\in I} |f_n(x) - f(x)| < \varepsilon$$

that is

$$\forall \varepsilon > 0 \ , \ \exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 \Longrightarrow \forall x \in I \ , |f_n(x) - f(x)| < \varepsilon,$$

which is exactly the definition of uniform convergence of (f_n) to f on I (and this definition implies that $f_n - f$ is bounded for $n \geq n_0$). □

Proposition 2.0.17 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on an interval I, taking values in $\mathbb{K}$, which converges* uniformly *on I to a function $f\colon I \longrightarrow \mathbb{K}$.*

If the f_n are bounded on I, then f is bounded on I.

Proof: We apply the definition of uniform convergence, with for example $\varepsilon = 1$; this gives

$$\exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \ , \quad \forall x \in I \ , \ |f_n(x) - f(x)| \leq 1 \, .$$

We then have, in particular:

$$\forall x \in I \ , \ |f(x) - f_{n_0}(x)| \leq 1$$

hence, using the triangle inequality

$$\forall x \in I \ , \ |f(x)| \leq |f_{n_0}(x)| + 1 \leq \|f_{n_0}\|_\infty^I + 1$$

which shows that f is bounded on I. □

Remark 2.0.18 *The result does not hold if there is only pointwise convergence: consider for example the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ by:*

$$f_n(x) = \frac{n}{nx+1} \text{ if } x \in]0,1] \quad \text{and} \quad f_n(0) = 0 \, .$$

Remark 2.0.19 *The set $\mathscr{B}(I,\mathbb{K})$ of bounded functions from I to $\mathbb{K}$ is a normed vector space for the norm defined by*

$$\forall f \in \mathscr{B}(I,\mathbb{K}), \quad \|f\|_\infty^I = \sup_{x\in I} |f(x)|$$

This norm is called the uniform convergence norm.

By Proposition 2.0.17, if (f_n) is a sequence of elements of $\mathscr{B}(I,\mathbb{K})$ which converges uniformly to $f \in \mathcal{A}(I,\mathbb{K})$, then $f \in \mathscr{B}(I,\mathbb{K})$, and the uniform convergence of (f_n) to f is then written

$$\lim_{n\to\infty} \|f_n - f\|_\infty^I = 0$$

that is, the sequence (f_n) tends to f in the normed vector space $\Big(\mathscr{B}(I,\mathbb{K}), \|.\|_\infty\Big)$.

Examples 2.0.20

1. *Let, for $n \in \mathbb{N}$,*

$$f_n(x) = \begin{cases} [0,1] \to \mathbb{R} \\ \quad x \mapsto x^n \end{cases}$$

   Then the sequence (f_n) converges pointwise on $[0,1]$ to the function

$$f : x \longmapsto \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x = 1 \end{cases}$$

   However, $\|f_n - f\|_\infty^{[0,1[} = \sup_{x\in[0,1[} |f_n(x) - f(x)| = \sup_{x\in[0,1[} x^n = 1$, so the sequence (f_n) does not converge uniformly to f on $[0,1]$.
   However, there is uniform convergence on any segment of the form $[0,a]$ with $0 \leq a < 1$, since $\|f_n - f\|_\infty^{[0,a]} = \sup_{x\in[0,a]} |f_n(x) - f(x)| = a^n$ tends to 0 as $n \longrightarrow +\infty$.

2. *Let, for $n \in \mathbb{N}^*$,*

$$f_n(x) = \begin{cases} [0,1] \to \mathbb{R} \\ \quad x \mapsto \dfrac{nx}{1+nx} \end{cases}$$

   If $x = 0$, $f_n(0) = 0$ for all $n \in \mathbb{N}^*$, and otherwise, $\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{nx}{nx} = 1$ so the sequence (f_n) converges pointwise on $[0,1]$ to the function

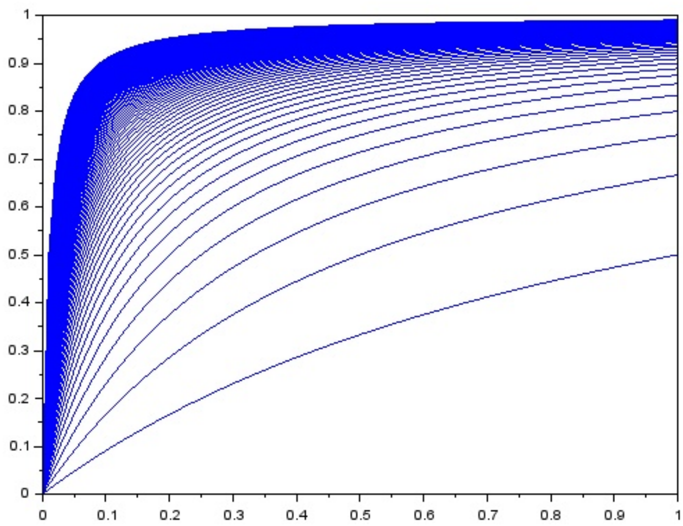


Figure 2.4: The Sequence functions $\frac{nx}{1+nx}$

$$f : x \longmapsto \begin{cases} 1 & \text{if } x \in]0,1] \\ 0 & \text{if } x = 0 \end{cases}.$$

However, $\|f_n - f\|_\infty^{[0,1]} = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in]0,1]} \frac{1}{1+nx} = 1$, *so the sequence* (f_n) *does not converge uniformly to* f *on* $[0,1]$.

However, there is uniform convergence on any segment of the form $[a,1]$ *with* $0 < a \le 1$, *since* $\|f_n - f\|_\infty^{[a,1]} = \sup_{x\in[a,1]} |f_n(x) - f(x)| = \frac{1}{1+na}$ *tends to* 0 *as* $n \longrightarrow +\infty$.

3. *Let, for* $n \in \mathbb{N}^*$,

$$f_n(x) = \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \min(n, \dfrac{x^2}{n}). \end{cases}$$

The sequence (f_n) *converges pointwise on* $\mathbb{R}$ *to the zero function. Indeed:*

Let $x \in \mathbb{R}$, *be fixed. There exists an integer* n_0 *such that, for all* $n \ge n_0$, *we have* $\frac{x^2}{n} < n$ *therefore, for* $n \ge n_0$, $f_n(x) = \frac{x^2}{n}$ *hence* $\lim_{n\to+\infty} f_n(x) = 0$.

However, there is no uniform convergence on $\mathbb{R}$, *since* $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} f_n(x) = n$.

However, there is uniform convergence on any segment $[-a,a]$ $(a > 0)$. *Indeed, from a certain rank* n_0, *we have* $\frac{a^2}{n} < n$, *so for all* $x \in [-a,a]$, *we will have, for* $n \ge n_0$, $f_n(x) = \frac{x^2}{n}$ *and* $\sup_{x\in[-a,a]} |f_n(x) - f(x)| = \frac{a^2}{n}$, *which tends to* 0 *as* n *tends to* $+\infty$.

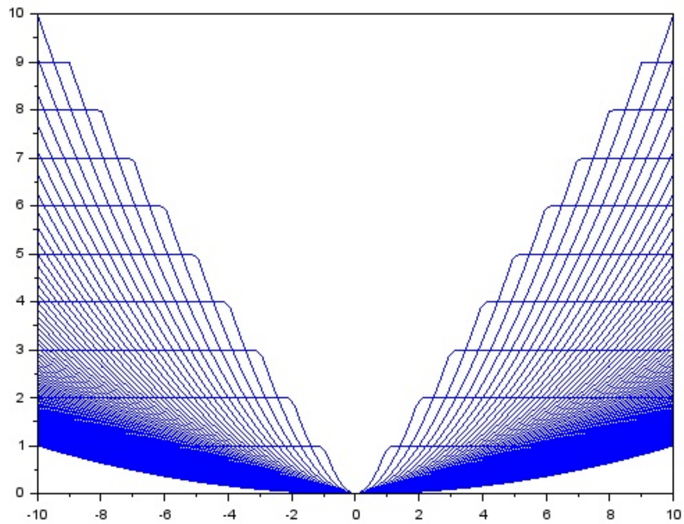


Figure 2.5: The Sequence functions $\min(n, \frac{x^2}{n})$

4. *Let, for* $n \in \mathbb{N}^*$,

$$f_n(x) = \begin{cases} [0,1] \to \mathbb{R} \\ \quad x \mapsto \dfrac{x+n}{n+4nx^2}. \end{cases}$$

The sequence (f_n) *converges pointwise on* $[0,1]$ *to the function* $f\colon x \longmapsto \dfrac{1}{1+4x^2}$ *Here there is uniform convergence on* $[0,1]$ *because:*

$\forall x \in [0,1],\ f_n(x) - f(x) = \dfrac{x}{n(1+4x^2)}$ *therefore* $\|f_n - f\|_\infty^{[0,1]} = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| \leq \dfrac{1}{n}$.

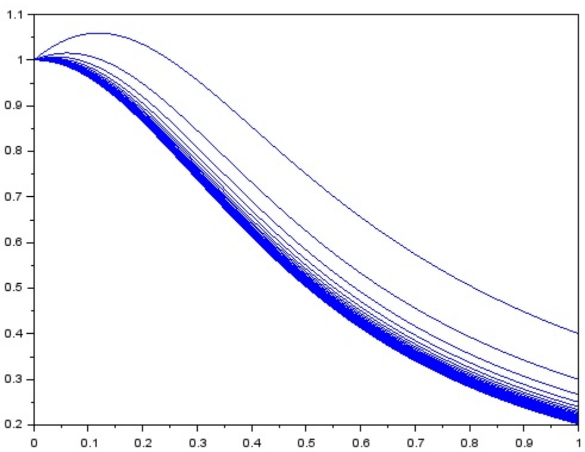


Figure 2.6: The Sequence functions $\frac{x+n}{n+4nx^2}$

2.0.3 Dini's Theorem for Sequences of Functions

The following result, known as Dini's theorem, characterizes a class of function sequences for which pointwise convergence ensures uniform convergence

Theorem 2.0.21 (Dini's Theorem) *Assume that:*

1. $f_n \in \mathcal{C}(I, \mathbb{R})$ $\forall n \in \mathbb{N}$, *where* I *is a closed bounded interval of* $\mathbb{R}$.
2. $(f_n)_n$ *converges pointwise to a continuous function* f *on* I.
3. $(f_n)_n$ *is a monotone sequence.*

Then $(f_n)_n$ *converges uniformly to* f *on* I.

Proof: Suppose $(f_n)_n$ is increasing, i.e.,

$$f_{n+1} \geq f_n \quad \forall n \in \mathbb{N}.$$

Let, for $n \in \mathbb{N}$

$$\delta_n := f - f_n \geq 0$$

since $f_n \underset{n \to +\infty}{\longrightarrow} f$. Then $(\delta_n)_n$ is a decreasing sequence of continuous functions converging pointwise to 0. We denote

$$\alpha_n := \sup_{x \in I} \delta_n(x).$$

To show the uniform convergence of $(f_n)_n$ to f, it suffices to show that $(\alpha_n)_n$ converges to 0. It is clear that $(\alpha_n)_n$ is decreasing and positive. It is therefore convergent to a real number $\alpha \geq 0$. We now show that $\alpha = 0$. For this, we will argue by contradiction, i.e., we assume that $\alpha > 0$. We consider, for $n \in \mathbb{N}$, the sets

$$K_n = \left\{ x \in I, \delta_n \geq \frac{\alpha}{2} \right\}.$$

Each set K_n is non-empty, bounded because contained in I, and as the preimage of the closed set $[\frac{\alpha}{2}, +\infty[$ by the continuous function δ_n, K_n is closed in I therefore in $\mathbb{R}$. As $\delta_{n+1} \geq \delta_n$, we have $K_{n+1} \subset K_n$. It follows,

$$\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset.$$

Therefore, there exists c belonging to K_n $\forall n$, and consequently $\delta_n(c) \geq \dfrac{\alpha}{2}$ for all n, which contradicts the pointwise convergence to 0. □

Example 2.0.22 *Let* $I = [-1, 1]$ *and* $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ *for* $n \in \mathbb{N}^*$. *We will satisfy the assumptions of Dini's theorem.*

1. *It is clear that* f_n *is continuous on* I, $\forall n \in \mathbb{N}^*$
2. $(f_n)_n$ *is decreasing since for* $x \in [-1, 1]$, *we have*

$$f_{n+1}(x) = \sqrt{x^2 + \frac{1}{(1+n)^2}} \leq \sqrt{x^2 + \frac{1}{n^2}} = f_n(x).$$

3. f_n converges pointwise on I to the continuous function $f(x) = |x|$.

Applying Dini's theorem, we conclude that f_n converges uniformly to the function f on $[-1, 1]$.

Remark 2.0.23 ***Caution****. Dini's theorem does not hold if we do not assume that I is a closed bounded interval of $\mathbb{R}$, as the following example shows:*

Example 2.0.24 *For $n \in \mathbb{N}$, we define f_n on $I =]0, 1[$ by*

$$f_n(x) = \frac{-1}{1 + nx}.$$

This sequence is increasing on I. Indeed, for all $x \in I$, we have

$$f_{n+1}(x) = \frac{-1}{1 + (n+1)x} \geq \frac{-1}{1 + nx} = f_n(x).$$

Moreover, $(f_n)_n$ converges pointwise to $f = 0$ on I. But $(f_n)_n$ does not converge uniformly to $f = 0$ on I, because

$$f_n(\frac{1}{n}) = \frac{-1}{2} \text{ and } \lim_{n \to +\infty} \left| f_n(\frac{1}{n}) - f(\frac{1}{n}) \right| = \frac{1}{2}.$$

Theorem 2.0.25 (Cauchy condition for uniform convergence) *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions defined on an interval I, taking values in $\mathbb{K}$.*

1. *If the sequence (f_n) converges uniformly to a function $f \colon I \to \mathbb{K}$, it satisfies*
   $$(*) \quad \forall \varepsilon > 0, \ \exists \underbrace{n_0}_{\text{only depends on } \varepsilon} \in \mathbb{N}, \text{ such that } \forall x \in I, \ \forall p, q \geq n_0, \ |f_p(x) - f_q(x)| < \varepsilon$$

2. If E is complete, *and if (f_n) satisfies the uniform Cauchy criterion $(*)$, then there exists a function $f \colon A \to E$ such that (f_n) converges uniformly to f on A.*

Proof:

1. If (f_n) UC to f, we have

   $$\forall \varepsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n \in \mathbb{N}, \ n \geq n_0 \Longrightarrow \ \forall x \in A, \ \|f_n(x) - f(x)\| < \frac{\varepsilon}{2}$$

   so, using the triangle inequality, we will have, for $p, q \geq n_0$ and for all $x \in A$:

   $$\|f_p(x) - f_q(x)\| = \|f_p(x) - f(x)) + (f(x) - f_q(x)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

   which is the desired result.

2. Suppose that the sequence (f_n) satisfies $(*)$, then, for all $x \in A$, the sequence $(f_n(x))$ satisfies the Cauchy criterion in E. E being complete, this sequence converges to an element of E which we will denote $f(x)$.

   By then letting q tend to $+\infty$ in $(*)$, we obtain

   $$\forall \varepsilon > 0 \ , \ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \ , \ \forall x \in A \ , \ \|f_p(x) - f(x)\|_E < \varepsilon$$

   which is exactly the definition of uniform convergence of (f_n) to f.

□

Remark 2.0.26 *In what follows, the normed vector space E is assumed to be complete.*

2.1 Properties of Uniform Convergence

In this section, we prove that, unlike pointwise convergence, uniform convergence preserves both boundedness and continuity. However, uniform convergence does not preserve differentiability any better than pointwise convergence. Nevertheless, we present a result that allows us to differentiate a convergent sequence, with the key assumption being that the derivatives converge uniformly.

2.1.1 Continuity of the Limit of a Sequence of Functions

One of the most important properties of uniform conver- gence is that it preserves continuity. The following theorem is now *assumed*:

Theorem 2.1.1 (Interchanging Limits (or Double Limit Theorem)) *Let (f_n) be a sequence of functions from I to $\mathbb{K}$, which converges* uniformly *on I to a function $f\colon I \to \mathbb{K}$.*

Let $a \in \bar{I}$ (possibly $\pm\infty$). Assume that, for any integer n (at least from a certain rank), the limit $\lim\limits_{\substack{x\to a\\x\in I}} f_n(x) = \ell_n$ exists.

Then the sequence $(\ell_n)_{n\in\mathbb{N}}$ converges to an element $\ell \in \mathbb{K}$, and moreover:

$$\lim_{\substack{x\to a\\x\in I}} f(x) = \ell.$$

In short: $\lim\limits_{x\to a}\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\lim\limits_{x\to a} f_n(x)$.

Remark 2.1.2 *Uniform convergence is essential here, as shown by the example of the sequence* $\big(x \mapsto x^n\big)$ *on* $[0,1]$.

Proof:

- Since (f_n) converges uniformly to f , we have:

  $$\forall\varepsilon > 0\ ,\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0\ ,\ \forall x \in I\ ,\ |f_n(x) - f(x)| \leq \frac{\varepsilon}{2}\,.$$

  Using the triangle inequality, we deduce:

  $$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \text{such that } \forall x \in I,\ \forall p, q \geq n_0,\ |f_p(x) - f_q(x)| < \varepsilon$$

  By letting x tend to a in this property, we obtain

  $$\forall\varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \text{such that } \forall p, q \geq n_0,\ |\ell_p - \ell_q| \leq \varepsilon\,.$$

  This means that the sequence (ℓ_n) is a Cauchy sequence in $\mathbb{K}$ (complete), therefore converges to an element ℓ of $\mathbb{K}$.

- Since (f_n) converges uniformly to f , we have, if we are given $\varepsilon > 0$:

  $$\exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0\ ,\ \forall x \in I\ ,\ |f_n(x) - f(x)| \leq \frac{\varepsilon}{3}\,.$$

  By writing the definition of $\lim\limits_{n\to+\infty} \ell_n = \ell$, we also have:

  $$\exists n_1 \in \mathbb{N} \text{ such that } \forall n \geq n_1\ ,\ |\ell_n - \ell| \leq \frac{\varepsilon}{3}$$

  Let us then choose $N \geq \max(n_0, n_1)$. The definition of $\lim\limits_{\substack{x\to a\\x\in I}} f_N(x) = \ell_N$ is written:

  $$\exists V \in \mathcal{V}(a) \text{ such that } \forall x \in V \cap I\ ,\ |f_N(x) - \ell_N| \leq \frac{\varepsilon}{3}\,.$$

We will then have, for all $x \in V \cap I$, using the triangle inequality:

$$|f(x) - \ell| = \big|\big(f(x) - f_N(x)\big) + \big(f_N(x) - \ell_N\big) + \big(\ell_N - \ell\big)\big| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which is exactly the definition of $\lim\limits_{\substack{x \to a \\ x \in I}} f(x) = \ell$.

□

The concept of uniform convergence is crucial for preserving the stability of continuity, Riemann integration, and differentiation properties of the limit function. We begin by discussing the continuity of uniform limits.
The following theorem shows that uniform convergence preserves the continuity of the limit.

Theorem 2.1.3 (Continuity of the Limit) *Let (f_n) be a sequence of functions from I to $\mathbb{K}$, which converges pointwise to a function $f : I \to \mathbb{K}$.*
Let $a \in I$. Assume that:

- *the f_n are continuous at a (at least from a certain rank);*
- *there exists a neighborhood V of a such that* the sequence (f_n) converges uniformly to f on V.

Then f is continuous at a.

Proof: Let $\varepsilon > 0$. By definition of uniform convergence, we have in particular:

$$\exists N \in \mathbb{N} \text{ such that } \ \forall x \in V, \ |f_N(x) - f(x)| < \frac{\varepsilon}{3}.$$

Since f_N is continuous at a we have :

$$\exists V' \in \mathcal{V}(a) \text{ such that } \ \forall x \in V', \ |f_N(x) - f_N(a)| < \frac{\varepsilon}{3}$$

Therefore, for all $x \in V \cap V'$ we will have, using the triangle inequality:

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| < \varepsilon$$

which is the definition of the continuity of f at a. □

Definition 2.1.4 *If the sequence (f_n) converges pointwise to f on I and if, for all $a \in I$ there exists a neighborhood V of a such that the convergence of (f_n) to f on V is uniform, we will say that there is local uniform convergence on I.*

Remark 2.1.5 *It is clear that, if there is uniform convergence on the whole of I, then there is a fortiori local uniform convergence; the converse is false, as shown by the example of the sequence of functions $(x \mapsto x^n)$ on $[0, 1[$.*

Corollary 2.1.6 *If the sequence (f_n) converges pointwise to f on I,* the convergence being local uniform*, and if the f_n are continuous on I, then f is continuous on I.*

Proof: Indeed, for all $a \in I$ there exists a neighborhood V of a such that the sequence (f_n) converges uniformly to f on V. According to the previous theorem, f is continuous at a, and since this is true for all $a \in I$, f is continuous on I □

A stronger result implies a weaker one:

Corollary 2.1.7 *If the sequence (f_n) converges uniformly to f on I, and if the f_n are continuous on I, then f is continuous on I.*

Remark 2.1.8 *This theorem can sometimes be used to show that there is* no *uniform convergence.*

Let's take the first example from the chapter, with $f_n(x) = x^n$ for $x \in [0,1]$. We saw that the sequence (f_n) converges pointwise on $[0,1]$ to the function

$$f\colon x \longmapsto \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x = 1 \end{cases}$$

The f_n are continuous on $[0,1]$ but not f: there cannot therefore be uniform convergence on $[0,1]$.

Example 2.1.9 *Consider the sequence of functions $(f_n)_n$ defined by:*

$$f_n(x) = \frac{x\sqrt{n}}{1+nx^2} \quad \text{where} \quad x \in [0,1].$$

It is clear $(f_n)_n$ converges pointwise to f on $[0,1]$ with

$$f(x) = 0 \quad \forall x \in [0,1].$$

Moreover, f is continuous on $[0,1]$, but $(f_n)_n$ does not converge uniformly to f on $[0,1]$. Indeed, we have

$$\delta_n(x) = |f_n(x) - f(x)| = \left| \frac{x\sqrt{n}}{1+nx^2} - 0 \right| = \frac{x\sqrt{n}}{1+nx^2}.$$

By differentiating δ_n, we obtain

$$\delta_n'(x) = \frac{\sqrt{n}(1-nx^2)}{(1+nx^2)^2}.$$

The function δ_n has a maximum at $x_n = \frac{1}{\sqrt{n}}$, therefore

$$\sup_{x\in[0,1]} \delta_n(x) = \delta_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2} \neq 0.$$

Convergence is not uniform.

2.1.2 Integration of a Sequence of Functions on a Segment

The following theorem allows us to interchange the limit with the integral sign.

Theorem 2.1.10 (Interchanging Limit-Integral on a Segment) *Let (f_n) be a sequence of continuous functions on a segment $[a,b]$ of $\mathbb{R}$, and* converging uniformly on *$[a,b]$ to a function f.*
Then f is continuous on $[a,b]$ and

$$\int_a^b f(t)dt = \lim_{n\to+\infty} \int_a^b f_n(t)dt.$$

Proof: The continuity of f is ensured by Theorem 2.1.3. The continuity of the functions involved also ensures the existence of the integrals considered. We then have

$$\begin{aligned} \left| \int_a^b f_n(t)dt - \int_a^b f(t)dt \right| = \left| \int_a^b (f_n(t) - f(t))dt \right| &\leq \int_a^b |f_n(t) - f(t)|\, dt \\ &\leq \int_a^b \|f_n - f\|_\infty\, dt = (b-a)\, \|f_n - f\|_\infty \end{aligned}$$

and the result follows from $\lim\limits_{n\to+\infty} \|f_n - f\|_\infty = 0$. □

Remark 2.1.11 *The theorem also applies to a sequence of functions* f_n piecewise continuous, *which converges uniformly on* $[a,b]$ *to a function* f piecewise continuous. *The proof is similar,* but we must also check the piecewise continuity of f, which is no longer ensured by uniform convergence.

Remark 2.1.12 *The two assumptions « uniform convergence » and « the integration interval is a segment » are essential, as shown by the following examples.*

1. *Let* $(f_n)_{n\geq 1}$ *be the sequence of functions defined on* $[0,1]$ *by*

$$f_n(0) = f_n(\frac{1}{n}) = f_n(1) = 0 \quad ; \quad f_n(\frac{1}{2n}) = n \quad \textit{and} \quad f_n \textit{ continuous piecewise affine.}$$

   We have already shown that the sequence (f_n) *converges pointwise on* $[0,1]$ *to the zero function. However, for all* $n \in \mathbb{N}^*$, $\displaystyle\int_0^1 f_n(t)dt = \frac{1}{2}$ *does not converge to* 0*!*

2. *Let* $(f_n)_{n\geq 2}$ *be the sequence of functions defined by:*

$$f_n(t) = \frac{1}{n} \textit{ for } t \in [0, n - \frac{1}{n}];\ f_n(t) = 0 \textit{ for } t \geq n \textit{ and } f_n \ \textit{ continuous piecewise affine.}$$

   Then $\|f_n\|_\infty^{\mathbb{R}_+} = \dfrac{1}{n}$ *so the sequence* (f_n) *converges uniformly on* $\mathbb{R}_+$ *to the zero function. However, it is easy to check that* $\displaystyle\lim_{n\to+\infty} \int_{\mathbb{R}_+} f_n = 1.$

Exercise 2.1.13 *Let* $f_n \colon [0,1] \to \mathbb{R}$ *be defined by*

$$f_n(x) = n^2 x(1 - nx) \textit{ if } x \in [0, \frac{1}{n}] \quad \textit{and} \quad f_n(x) = 0 \textit{ otherwise.}$$

1. *Study the pointwise limit of the sequence* (f_n).
2. *Compute* $\displaystyle\int_0^1 f_n(t)\,\mathrm{d}t$. *Is there uniform convergence of the sequence* (f_n) *on* $[0,1]$*?*
3. *Study the uniform convergence on* $[a,1]$ *with* $a > 0$.

Solution 2.1.14

1. *For* $x = 0$, $f_n(x) = 0$, *and for* $x > 0$, *we also have* $f_n(x) = 0$ *for large enough* n *(as soon as* $x > \frac{1}{n}$*). Hence, the sequence* (f_n) *converges pointwise (PWC) to the zero function on* $[0,1]$.
2. $\displaystyle\int_0^1 f_n(t)\,\mathrm{d}t = \int_0^{\frac{1}{n}} n^2 t(1 - nt)\,\mathrm{d}t = \int_0^1 u(1 - u)\,\mathrm{d}u = \frac{1}{6}.$

   Therefore, there is no uniform convergence of the sequence (f_n)*, by virtue of the theorem on the interchange of limit and integral.*
3. *For large enough* n *(as soon as* $a > \frac{1}{n}$*),* $\displaystyle\sup_{x\in[a,1]} |f_n(x)| = 0$*, hence the sequence* (f_n) *converges uniformly (UC) to 0 on* $[a,1]$.

2.1.3 Differentiation of a Sequence of Functions

The uniform convergence of differentiable functions does not, in general, imply anything about the convergence of their derivatives or the differentiability of their limit. As noted above, this is because the values of two functions may be close together while the values of their derivatives are far apart (if, for example, one function varies slowly while the other oscillates rapidly, as in Examples 4.5 and 4.6). Thus, we have to impose strong conditions on a sequence of functions and their derivatives if we hope to prove that $f_n \to f$ implies $f'_n \to f'$.
Indeed, let (f_n) be a sequence of functions of class C^1, converging pointwise on an interval I to a function f of class C^1.
We do not necessarily have $(\lim f_n)' = \lim f'_n$, even if there is uniform convergence

Example 2.1.15 *Let* $f_n : x \in \mathbb{R} \mapsto \dfrac{\sin nx}{\sqrt{n}}$ *for* $n \in \mathbb{N}^*$.
Then $\|f_n\|_\infty^{\mathbb{R}} = \dfrac{1}{\sqrt{n}}$, *so the sequence* (f_n) *converges uniformly on* $\mathbb{R}$ *to the zero function.*
However, $f'_n(x) = \sqrt{n}\cos nx$, *and the sequence* (f'_n) *does not even have a pointwise limit!*

Example 2.1.16 *Consider the sequence* (f_n) *of functions* $f_n : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_n(x) = \frac{x}{1+nx^2}.$$

Then $f_n \to 0$ *uniformly on* $\mathbb{R}$. *To show this, we write*

$$|f_n(x)| = \frac{1}{\sqrt{n}}\left(\frac{\sqrt{n}|x|}{1+nx^2}\right) = \frac{1}{\sqrt{n}}\frac{t}{1+t^2}$$

where $t = \sqrt{n}|x|$. *We have*

$$\frac{t}{1+t^2} \le \frac{1}{2} \quad \text{for all } t \in \mathbb{R},$$

since $(1-t)^2 \ge 0$, *which implies that* $2t \le 1+t^2$. *Using this inequality, we get*

$$|f_n(x)| \le \frac{1}{2\sqrt{n}} \quad \text{for all } x \in \mathbb{R}.$$

Hence, given $\varepsilon > 0$, *choose* $N = 1/(4\varepsilon^2)$. *Then*

$$|f_n(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R} \text{ if } n > N,$$

which proves that (f_n) *converges uniformly to 0 on* $\mathbb{R}$. *(Alternatively, we can obtain the same result by using calculus to find the maximum value of* $|f_n|$ *on* $\mathbb{R}$.*)*

Each f_n *is differentiable with*

$$f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2}.$$

It follows that $f'_n \to g$ *pointwise as* $n \to \infty$, *where*

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

The convergence is not uniform since g *is discontinuous at 0. Thus,* $f_n \to 0$ *uniformly, but* $f'_n(0) \to 1$, *so the limit of the derivatives is not the derivative of the limit.*

Therefore, additional assumptions are required to differentiate the limit of a sequence of functions.

Theorem 2.1.17 (Differentiation of the Limit of a Sequence of Functions) *Suppose that* (f_n) *is a sequence of differentiable functions* $f_n : (a,b) \to \mathbb{R}$ *such that* $f_n \to f$ *pointwise and* $f'_n \to g$ *uniformly for some* $f, g : (a,b) \to \mathbb{R}$. *Then* f *is differentiable on* (a,b) *and* $f' = g$.

Proof: Let $c \in (a,b)$, and let $\varepsilon > 0$ be given. To prove that $f'(c) = g(c)$, we estimate the difference quotient of f in terms of the difference quotients of the f_n :

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + \left|f_n'(c) - g(c)\right|$$

where $x \in (a,b)$ and $x \neq c$. We want to make each of the terms on the right-hand side of the inequality less than $\varepsilon/3$. This is straightforward for the second term (since f_n is differentiable) and the third term (since $f_n' \to g$). To estimate the first term, we approximate f by f_m, use the mean value theorem, and let $m \to \infty$.

Since $f_m - f_n$ is differentiable, the mean value theorem implies that there exists ξ between c and x such that

$$\begin{aligned}\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} &= \frac{(f_m - f_n)(x) - (f_m - f_n)(c)}{x-c} \\ &= f_m'(\xi) - f_n'(\xi)\end{aligned}$$

Since (f_n') converges uniformly, it is uniformly Cauchy by Theorem 2.0.25. Therefore there exists $N_1 \in \mathbb{N}$ such that

$$\left|f_m'(\xi) - f_n'(\xi)\right| < \frac{\varepsilon}{3} \quad \text{for all } \xi \in (a,b) \text{ if } m,n > N_1$$

which implies that

$$\left|\frac{f_m(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| < \frac{\varepsilon}{3}$$

Taking the limit of this equation as $m \to \infty$, and using the pointwise convergence of (f_m) to f, we get that

$$\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| \leq \frac{\varepsilon}{3} \quad \text{for } n > N_1$$

Next, since (f_n') converges to g, there exists $N_2 \in \mathbb{N}$ such that

$$\left|f_n'(c) - g(c)\right| < \frac{\varepsilon}{3} \quad \text{for all } n > N_2$$

Choose some $n > \max(N_1, N_2)$. Then the differentiability of f_n implies that there exists $\delta > 0$ such that

$$\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| < \frac{\varepsilon}{3} \quad \text{if } 0 < |x-c| < \delta$$

Putting these inequalities together, we get that

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \varepsilon \quad \text{if } 0 < |x-c| < \delta$$

which proves that f is differentiable at c with $f'(c) = g(c)$. □ Like Theorem 2.1.3, Theorem 2.1.17 can be interpreted as giving sufficient conditions for an exchange in the order of limits:

$$\lim_{n\to\infty} \lim_{x\to c} \left[\frac{f_n(x)-f_n(c)}{x-c}\right] = \lim_{x\to c} \lim_{n\to\infty} \left[\frac{f_n(x)-f_n(c)}{x-c}\right]$$

It is worth noting that in Theorem 2.1.17 the derivatives f_n' are not assumed to be continuous. If they are continuous, then one can use Riemann integration and the fundamental theorem of calculus to give a simpler proof (see Theorem 3.4.3).

Theorem 2.1.18 *Let* (f_n) *be a sequence of functions* of class C^1 *on an interval* I *of* $\mathbb{R}$*, taking values in* $\mathbb{K}$*. Assume that:*

a) The sequence of functions (f_n) *converges pointwise on* I *to a function* f*.*

b) The sequence of functions (f_n') *converges pointwise on* I *to a function* g*,* the convergence being local uniform on I .

Then the function f *is of class* C^1 *on* I*, and, for all* $x \in I$*,* $f'(x) = g(x)$ *(i.e., in short,* $(\lim f_n)' = \lim f_n'$*).*
Moreover, the sequence (f_n) *converges uniformly locally to* f*.*

Proof:

- Since the sequence of continuous functions (f_n') converges uniformly locally to g on I, according to Theorem 2.1.3 g is continuous on I.

- Let $a \in I$, and V be an interval containing a on which there is uniform convergence of the sequence (f_n') to g. For all $x \in V$ we have $f_n(x) = f_n(a) + \displaystyle\int_a^x f_n'(t)dt$. Since the convergence of the sequence (f_n') to g is uniform on the segment $[a, x]$ (or $[x, a]$), Theorem 3.4.8 gives: $\displaystyle\lim_{n\to+\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt$.

- Moreover, the pointwise convergence of the sequence (f_n) to f gives $\displaystyle\lim_{n\to+\infty} f_n(x) = f(x)$ and $\displaystyle\lim_{n\to+\infty} f_n(a) = f(a)$.

- We deduce, for all $x \in V$, $f(x) = f(a) + \displaystyle\int_a^x g(t)dt$. Consequently, f is of class C^1 on V and $f' = g$. This being true in the neighborhood of any $a \in I$, it is true on I (the notions of continuity and differentiability are local notions).

- Finally, if a is an element of I and if J is a segment containing a on which the sequence (f_n') converges uniformly to g (there exists one by assumption), we will have, thanks to the triangle inequality and the mean value inequality:

$$\begin{aligned}\forall x \in J \ , \ |f_n(x) - f(x)| &= \left|(f_n(a) - f(a)) + \left(\int_a^x \big(f_n'(t) - g(t)\big)dt\right)\right| \\ &\leq |f_n(a) - f(a)| + \ell(J) \left\|f_n' - g\right\|_\infty^J\end{aligned}$$

  by denoting $\ell(J)$ the length of J.

  Thus,

$$\|f_n - f\|_\infty^J \leq |f_n(a) - f(a)| + \ell(J) \left\|f_n' - g\right\|_\infty^J ,$$

  and since the convergence of (f_n') to g is uniform on J, we have $\displaystyle\lim_{n\to+\infty} \left\|f_n' - g\right\|_\infty^J = 0$ hence $\displaystyle\lim_{n\to+\infty} \|f_n - f\|_\infty^J = 0$, i.e. the sequence (f_n) converges uniformly to f on J.
  There is therefore indeed local uniform convergence of (f_n) to f.

□

Corollary 2.1.19 (Sequences of Functions of Class C^k**,** $k \geq 1$**)** *Let* (f_n) *be a sequence of functions of class* C^k *(*$k \in \mathbb{N}^*$*) on an interval* I *of* $\mathbb{R}$*, taking values in* $\mathbb{K}$*. Assume that:*

a) For all $j \in [[0, k-1]]$*, the sequence of functions* $(f_n^{(j)})$ *converges pointwise on* I*;*

b) The sequence of functions $(f_n^{(k)})$ *converges pointwise on* I *to a function* g*,* the convergence being local uniform.

Then, the function $f = \lim_{n\to+\infty} f_n$ *is of class* C^k *on* I*, we have* $f^{(k)} = g$ *and for* $j \in [[0, k-1]]$*, each sequence* $(f_n^{(j)})$ *converges uniformly locally to* $f^{(j)}$*.*

Proof: The proof is naturally done by induction on k.

- For $k = 1$, it is the previous theorem.
- Assume the proposition is true at rank $k-1$ with $k \geq 2$. Then let $h_n = f_n^{(k-1)}$. The assumptions allow us to apply Theorem 3.4.3 to the sequence (h_n); we deduce that the sequence (h_n) converges uniformly on any segment of I, its limit h being of class C^1 on I and such that $h' = g$.

  By the induction hypothesis, f is of class C^{k-1} on I and $f^{(k-1)} = h$, each sequence $(f_n^{(j)})$ for $0 \leq j \leq k-2$ converging uniformly to $f^{(j)}$ on any segment included in I.

  Thus $f^{(k-1)} = h$ is of class C^1 i.e. f is of class C^k, with $f^{(k)} = h' = g$ and the sequence $(f_n^{(k-1)})$ converges uniformly locally to $h = f^{(k-1)}$ which establishes the result at order k and completes the induction.

□

Corollary 2.1.20 (Sequences of Functions of Class C^∞) *Let* (f_n) *be a sequence of functions of class* C^∞ *on an interval* I *of* $\mathbb{R}$*, taking values in* $\mathbb{K}$*. Assume that:*

a) For all $j \in \mathbb{N}$*, the sequence of functions* $(f_n^{(j)})$ *converges pointwise on* I*;*

b) There exists $p \in \mathbb{N}^*$ *such that, for all* $k \geq p$*, the sequence of functions* $(f_n^{(k)})$ *converges pointwise on* I*,* the convergence being local uniform.

Then, the function $f = \lim_{n\to+\infty} f_n$ *is of class* C^∞ *on* I*, and for* $j \in \mathbb{N}$*, each sequence* $(f_n^{(j)})$ *converges uniformly locally to* $f^{(j)}$*.*

Proof: We apply the previous corollary to any order $k \geq p$. □

2.2 Series of Functions

A series of functions is useful in solving ordinary differential equations or partial differential equations. Very often, these equations do not have an obvious solution expressible using standard functions. Therefore, the idea is to look for solutions in the form of series.

2.2.1 Preliminaries

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions from an interval I to $\mathbb{K}$. We can then consider the sequence of functions $(S_n)_{n\in\mathbb{N}}$ defined by

$$\forall x \in I \ , \ S_n(x) = \sum_{k=0}^{n} u_k(x) \, .$$

To study the series of functions $\sum_{n\in\mathbb{N}} u_n$, is to study the sequence of functions (S_n).

2.2.2 Domain of Convergence

Definition 2.2.1 *The set D defined by*

$$D = \left\{ x \in I \ / \ \sum_{n\geq 0} u_n(x) \ \text{converges} \right\}$$

is called the domain of convergence of the series $\sum_{n\geq 0} u_n(x)$.

Examples 2.2.2 *We will calculate the domain of convergence for each series in the following series:*

1. *Let the series of functions $\sum_{n\geq 0} \frac{x^n}{n!}$ where $x \in \mathbb{R}$. Then, by d'Alembert's rule*

$$l = \lim_{n\to+\infty} \left| \frac{x^{n+1}}{(n+1)!} . \frac{n!}{x^n} \right| = \lim_{n\to+\infty} \frac{|x|}{n+1} = 0 < 1 \ \forall x \in \mathbb{R}.$$

Consequently, $\sum_{n\geq 0} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. Hence

$$D = \left\{ x \in \mathbb{R} \ / \ \sum_{n\geq 0} \frac{x^n}{n!} \ \text{converges} \right\} = \mathbb{R}.$$

2. *Consider the series of functions $\sum_{n\geq 0} \frac{x^n}{n}$ where $x \in \mathbb{R}$. Applying d'Alembert's rule, we get*

$$l = \lim_{n\to+\infty} \left| \frac{x^{n+1}}{(n+1)} . \frac{n}{x^n} \right| = \lim_{n\to+\infty} |x| \frac{n}{n+1} = |x|.$$

Then, if $|x| < 1$, the series $\sum_{n\geq 0} \frac{x^n}{n}$ converges and if $|x| > 1$, it diverges. Finally, for the case $|x| = 1$ i.e., $x \pm 1$, we obtain two infinite series $\sum_{n\geq 0} \frac{1}{n}$ and $\sum_{n\geq 0} \frac{(-1)^n}{n}$ which are respectively divergent and convergent. Thus the domain of convergence is

$$D = \left\{ x \in \mathbb{R} \ / \ \sum_{n\geq 0} \frac{x^n}{n} \ \text{converges} \right\} = [-1, 1[.$$

3. *To determine the domain of convergence of the series of functions $\sum_{n\geq 1} nx^n$ where $x \in \mathbb{R}$.*

Let's analyze the convergence of this series by examining the general term $u_n = nx^n$.

Step 1*: Using the Root Test*

The root test is useful here. For a series $\sum u_n$, the root test states that the series converges when

$$\limsup_{n\to\infty} \sqrt[n]{|u_n|} < 1.$$

For our series, we have $u_n = nx^n$. *So,*

$$|u_n| = n|x|^n.$$

Taking the n-th root, we get:

$$\sqrt[n]{|u_n|} = \sqrt[n]{n} \cdot |x|.$$

As $n \to \infty$, $\sqrt[n]{n} \to 1$, *so*

$$\limsup_{n\to\infty} \sqrt[n]{|u_n|} = |x|.$$

Therefore, the series converges if $|x| < 1$.

Step 2: *Convergence on the Boundary* $|x| = 1$

We now check whether the series converges on the boundary points $x = 1$ *and* $x = -1$.

(a) *At* $x = 1$:

$$\sum_{n\geq 1} n \cdot 1^n = \sum_{n\geq 1} n,$$

which diverges, since the series $\sum n$ *grows without bound.*

(b) *At* $x = -1$:

$$\sum_{n\geq 1} n \cdot (-1)^n = \sum_{n\geq 1} (-1)^n n,$$

which is an ***alternating series***. *However, the terms* $|n|$ *do not tend to zero, and instead increase in magnitude, so this series also diverges.*

Conclusion

The series $\sum_{n\geq 1} nx^n$ *converges for* $|x| < 1$ *and diverges for* $|x| \geq 1$. *Therefore, the **domain of convergence** is*

$$|x| < 1.$$

Consequently,

$$D = \left\{ x \in \mathbb{R} \ / \ \sum_{n\geq 1} nx^n \ \text{converges} \right\} =]-1, 1[.$$

Similarly, for series of functions, we define the notions of convergence that we studied previously

Definition 2.2.3 *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of functions defined on an interval* I *and taking values in* $\mathbb{K}$.

We say that the series of functions $\sum_{n\in\mathbb{N}} u_n$ **converges pointwise* on* I *if there exists a function* $S : I \to \mathbb{K}$ *such that the sequence of partial sums* (S_n) *converges pointwise on* I *to* S. *This means that, for all* $x \in I$, *the series* $\sum_{n\in\mathbb{N}} u_n(x)$, *taking values in* $\mathbb{K}$, *converges and that* $S(x) = \sum_{n=0}^{+\infty} u_n(x)$. S *is then called the *sum* of the series of functions* $\sum_{n\in\mathbb{N}} u_n$. *We also define the *remainder of order n**, $R_n = S - S_n = \sum_{k=n+1}^{+\infty} u_k$. *The sequence of functions* (R_n) *converges pointwise on* I *to the zero function.*

Definition 2.2.4 *We say that the series of functions* $\sum_{n\in\mathbb{N}} u_n$ *converges uniformly on* I *if there exists a function* $S\colon I \to \mathbb{K}$ *such that the sequence of functions* (S_n) *converges uniformly on* I *to* S*.*

Theorem 2.2.5 *The series of functions* $\sum_{n\in\mathbb{N}} u_n$ *converges uniformly on* I *if and only if it converges pointwise on* I *and the sequence of remainders* (R_n) *converges uniformly on* I *to the zero function (in other words,* $\lim_{n\to+\infty} \|R_n\|_\infty^I = 0$*).*

Proof: Indeed, if the series converges uniformly on I, it also converges pointwise. We can then define its sum $S : I \to E$.
By definition of the uniform convergence of the sequence (S_n) of partial sums to S, the sequence (R_n) converges uniformly on I to the zero function, because

$$\lim_{n\to+\infty} \|R_n\|_\infty^I = \lim_{n\to+\infty} \|S - S_n\|_\infty^I = 0.$$

The converse is identical. □

Remark 2.2.6 *As with sequences, we define the notion of* local uniform convergence *in the same way: it occurs when there is uniform convergence in the neighborhood of every point in* I*. Specifically, for all* $a \in I$*, there exists a neighborhood* V *of* a *such that* $\lim_{n\to+\infty} \|R_n\|_\infty^V = 0$*.*

Examples 2.2.7

1. *Study of the series of functions* $\sum_{n\geq 1} \frac{x^n}{n^2}$*.*

   - Pointwise Convergence :
     For $|x| > 1$*,* $\frac{x^n}{n^2}$ *does not tend to* 0 *as* $n \to +\infty$*, so the series* $\sum_{n\geq 1} \frac{x^n}{n^2}$ *diverges to infinity.*
     If $|x| \leq 1$*,* $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$ *so the series* $\sum_{n\geq 1} \frac{x^n}{n^2}$ *is absolutely convergent (therefore convergent) by comparison with the convergent series with positive terms* $\sum_{n\geq 1} \frac{1}{n^2}$*.*
     In conclusion, the series converges pointwise on $[-1, 1]$ *and we can therefore set:*

     $$\forall x \in [-1,1]\ ,\ f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2}\,.$$

   - Uniform Convergence :
     For all $x \in [-1, 1]$*, we have*

     $$|R_n(x)| = \left|\sum_{k=n+1}^{+\infty} \frac{x^k}{k^2}\right| \leq \sum_{k=n+1}^{+\infty} \left|\frac{x^k}{k^2}\right| \leq \sum_{k=n+1}^{+\infty} \frac{1}{k^2}$$

     so $\|R_n\|_\infty \leq \sum_{k=n+1}^{+\infty} \frac{1}{k^2}$ *and* $\lim_{n\to+\infty} \|R_n\|_\infty = 0$ *since* $\sum_{k=n+1}^{+\infty} \frac{1}{k^2}$ *is the remainder of a convergent infinite series.*

In conclusion, the series of functions $\sum_{n\geq 1} \frac{x^n}{n^2}$ *converges uniformly to* f *on* $[-1,1]$.

2. *Study of the series of functions* $\sum_{n\geq 0} \frac{x^2}{(x^2+1)^n}$.

- Pointwise Convergence :
  - *For all* $n \in \mathbb{N}$, $u_n(0) = 0$ *so* $\sum_{n=0}^{+\infty} u_n(0) = 0$.
  - *If* $x \neq 0$, *the infinite series* $\sum_{n\geq 0} u_n(x)$ *is a geometric series with common ratio* $\frac{1}{1+x^2} < 1$, *so it converges, and its sum* S *is such that* $S(x) = \frac{x^2}{1-\frac{1}{1+x^2}} = 1+x^2$.

  In conclusion, the series of functions converges pointwise on $\mathbb{R}$ *to the function*

$$S : x \mapsto \begin{cases} 0 & \textit{if } x = 0 \\ 1+x^2 & \textit{otherwise.} \end{cases}$$

- Uniform Convergence : *The* u_n *being continuous, so are the partial sums of the series; the limit function* S *not being continuous, there cannot be uniform convergence on* $\mathbb{R}$.
  However : *there is uniform convergence on any subset of* $\mathbb{R}$ *of the form* $A =]-\infty, -a] \cup [a, +\infty[$ *with* $a > 0$.
  Indeed, if $x \neq 0$,

$$R_n(x) = \sum_{k=n+1}^{+\infty} \frac{x^2}{(x^2+1)^k} = \frac{x^2}{(x^2+1)^{n+1}} \frac{1}{1-\frac{1}{x^2+1}} = \frac{1}{(x^2+1)^n}$$

  so $\|R_n\|_\infty^A = \frac{1}{(a^2+1)^n} \underset{n\to+\infty}{\longrightarrow} 0$.
  We deduce that there is local uniform convergence on $\mathbb{R}^*$.

As with infinite series, we have a necessary condition for uniform convergence. In fact, the contrapositive of this condition is very useful for establishing the divergence of a series of functions. This is given by the following proposition:

Proposition 2.2.8 *If the series of functions* $\sum_{n\in\mathbb{N}} u_n(x)$ *converges uniformly on* I, *then* $(u_n)_n$ *converges uniformly to 0 on* I.

Proof: Indeed, since $\sum_{n\in\mathbb{N}} u_n(x)$ converges uniformly on I i.e., $S_n \underset{n\longrightarrow+\infty}{\overset{c.u}{\rightarrow}} S$ on I, we therefore have

$$u_n = S_n - S_{n-1} \underset{n\longrightarrow+\infty}{\overset{c.u}{\rightarrow}} S - S = 0 \text{ on } I.$$

□

Consequently, we have the following corollary.

Corollary 2.2.9 *If* $u_n \underset{n\longrightarrow+\infty}{\overset{c.u}{\nrightarrow}} 0$ *on* I *i.e.* $\lim_{n\to+\infty} \sup_{x\in I} |u_n(x)| \neq 0$, *then the series* $\sum_{n\in\mathbb{N}} u_n(x)$ *does not converge uniformly on* I.

Example 2.2.10 *Consider the series of functions with general term u_n defined by*

$$u_n(x) = x^n \text{ where } x \in]-1, 1[= I.$$

Then, we have, for all $n \in \mathbb{N}$

$$\sup_{x \in I} |u_n(x)| = \sup_{x \in I} |x^n| = 1.$$

Hence, $\lim_{n \longrightarrow +\infty} \sup_{x \in I} |u_n(x)| \neq 0$. *Therefore* $\sum_{n \geq 0} x^n$ *does not converge uniformly on I.*

2.2.3 Dini's Theorem for Series of Functions

As with sequences of functions, Dini's theorem below shows that pointwise convergence implies uniform convergence under certain conditions.

Theorem 2.2.11 *Assume that:*

1. *for all $n \in \mathbb{N}$, $u_n \in C([a, b], \mathbb{R})$,*
2. *for all $x \in [a, b]$, $\forall n \in \mathbb{N}$, we have $u_n(x) \geq 0$ or $u_n(x) \leq 0$,*
3. $\sum_{n \geq 0} u_n$ *converges pointwise on $[a, b]$,*
4. $S = \sum_{n=0}^{+\infty} u_n$ *is continuous on $[a, b]$.*

Then, the convergence of the series $\sum_{n \geq 0} u_n$ *on $[a, b]$ is uniform.*

Proof: Let $S_n(x) = u_0(x) + u_1(x) + ... + u_n(x)$, $x \in [a, b]$. This sequence satisfies the assumptions of Dini's theorem for sequences of functions, i.e.,

1. $S_n \overset{c.s}{\underset{n \longrightarrow +\infty}{\rightarrow}} S$ on $[a, b]$, (pointwise convergence of $\sum_{n \geq 0} u_n$).
2. S_n and S are continuous (continuity of u_n) .
3. (S_n) is monotone (it is increasing if $u_n(x) \geq 0$ and decreasing if $u_n(x) \leq 0$). Then the convergence of S_n to S is uniform on $[a, b]$.

Which means that $\sum_{n \geq 0} u_n$ converges uniformly on $[a, b]$. □

2.2.4 Criterion for Uniform Convergence

Here we give two criteria for uniform convergence for series of functions.

2.2.5 Cauchy Criterion

Consider the series of functions $\sum_{n \geq 0} u_n$ where $u_n : I \longrightarrow \mathbb{R}$, $I \subset \mathbb{R}$.

The Cauchy condition for the uniform convergence of sequences immediately provides a corresponding Cauchy condition for the uniform convergence of series.

Theorem 2.2.12 *The series* $\sum_{n \geq 0} u_n$ *converges uniformly on I if and only if for any $\varepsilon > 0$,*

$$\exists n_0 \in \mathbb{N}, \forall n, m \in \mathbb{N}, \forall x \in I : \left(n, m \geq n_0 \Longrightarrow \left| \sum_{k=n+1}^{m} u_k(x) \right| < \varepsilon \right).$$

Proof: Let $(S_n)_n$ be the sequence of partial sums of the series $\sum_{n\geq 0} u_n$, then for all $x \in I$

$$S_n(x) = u_0(x) + u_1(x) + ... + u_n(x).$$

The series $\sum_{n\geq 0} u_n$ converges uniformly on I, i.e.,

$$S_n \underset{n\longrightarrow+\infty}{\overset{c.s}{\rightarrow}} S \text{ on } I. \tag{2.1}$$

Hence, by the Cauchy criterion for sequences of functions, (2.1) is equivalent to $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$,

$$\forall n, m \in \mathbb{N}, \forall x \in I \, : \, (n, m \geq n_0 \Longrightarrow |S_m - S_n| < \varepsilon) \, .$$

This is still equivalent to $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$,

$$\forall n, m \in \mathbb{N}, \forall x \in I \, : \, \left(n, m \geq n_0 \Longrightarrow \left| \sum_{k=n+1}^{m} u_k(x) \right| < \varepsilon \right).$$

□

2.2.6 Abel's Criterion

Consider the series of the form:

$$\sum_{n\geq 0} a_n(x)b_n(x) \quad \text{where} \quad a_n, b_n \, : \, I \subset \mathbb{R} \longrightarrow \mathbb{R}.$$

Thus, Abel's criterion is given by the proposition:

Proposition 2.2.13 *Assume that:*

1. *for $x \in I$, the sequence $(a_n(x))_{n\in\mathbb{N}}$ is decreasing with positive real values,*
2. *the sequence of functions $(a_n)_{n\in\mathbb{N}}$ converges uniformly to 0 on I,*
3. *the sequence of partial sums of $\sum_{n\geq 0} b_n$ is bounded, i.e.,*

$$\exists M > 0, \forall n \in \mathbb{N}, \forall x \in I \, : \, \left| \sum_{k=0}^{n} b_k(x) \right| \leq M.$$

Then, the series $\sum_{n\geq 0} a_n(x)b_n(x)$ converges uniformly on I.

Proof: The proof of this proposition is analogous to that for infinite series (see the previous chapter). □

2.2.7 Absolute Convergence

As with infinite series, we define the notion of absolute convergence for a series of functions.

Definition 2.2.14 *The series $\sum_{n\geq 0} u_n(x)$ is said to be absolutely convergent if $\sum_{n\geq 0} |u_n(x)|$ is convergent.*

Example 2.2.15 *Consider the series of functions*

$$\sum_{n\geq 1} \frac{1-x^n}{1+x^n}\frac{1}{n^2}, \quad x \in [0, +\infty[. \tag{2.2}$$

Let $x \geq 0$ be any fixed value, then we can write

$$\left|\frac{1-x^n}{1+x^n}\frac{1}{n^2}\right| \leq \frac{1}{n^2},$$

since we have

$$\left|\frac{1-x^n}{1+x^n}\right| \leq 1 \quad \forall x \geq 0.$$

And since the Riemann series $\sum_{n\geq 1}\frac{1}{n^2}$ is convergent, then by comparison, the series $\sum_{n\geq 1}\frac{1-x^n}{1+x^n}\frac{1}{n^2}$ is convergent. This shows that the series (2.2) is absolutely convergent.

2.2.8 Normal Convergence of a Series of Functions

We introduce an additional notion of convergence. It is very useful in practice for establishing the uniform convergence of a series of functions, because it reduces the study of the latter to that of the convergence of a series with positive terms.

Definition 2.2.16 *We say that the series of functions $\sum_{n\in\mathbb{N}} u_n$ converges normally on I if:*

- *the functions u_n are bounded on I (at least from a certain rank)*
- *and the infinite series $\sum_{n\geq 0} \|u_n\|_\infty^I$ is convergent (by denoting as usual: $\|u_n\|_\infty^I = \sup_{x\in I}|u_n(x)|$).*

Theorem 2.2.17 (Weierstrass M-test) *Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$, and suppose that for every $n \in \mathbb{N}$ there exists a constant $M_n \geq 0$ such that*

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \quad \sum_{n=1}^{\infty} M_n < \infty$$

Then

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on A.

Proof: The result follows immediately from the observation that $\sum f_n$ is uniformly Cauchy if $\sum M_n$ is Cauchy.

In detail, let $\varepsilon > 0$ be given. The Cauchy condition for the convergence of a real series implies that there exists $N \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon \quad \text{for all } n > m > N$$

Then for all $x \in A$ and all $n > m > N$, we have

$$\begin{aligned}\left|\sum_{k=m+1}^{n} f_k(x)\right| &\le \sum_{k=m+1}^{n} |f_k(x)| \\ &\le \sum_{k=m+1}^{n} M_k \\ &< \varepsilon\end{aligned}$$

Thus, $\sum f_n$ satisfies the uniform Cauchy condition in Theorem 2.2.12, so it converges uniformly. □

Example 2.2.18 *We consider the geometric series*

$$\sum_{n=0}^{\infty} x^n$$

If $|x| \le \rho$ where $0 \le \rho < 1$, then

$$|x^n| \le \rho^n, \quad \sum_{n=0}^{\infty} \rho^n < 1$$

The M-test, with $M_n = \rho^n$, implies that the series converges uniformly on $[-\rho, \rho]$.

Example 2.2.19 *The series*

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos\left(3^n x\right),$$

converges uniformly on $\mathbb{R}$ by the M-test since

$$\left|\frac{1}{2^n} \cos\left(3^n x\right)\right| \le \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Theorem 2.2.20 *If $(u_n)_{n\in\mathbb{N}}$ is a sequence of functions from I to $\mathbb{K}$ such that the series of functions $\sum_{n\in\mathbb{N}} u_n$ is* normally convergent *on I, then:*

1. *For all $x \in I$, the series $\sum_{n\in\mathbb{N}} u_n(x)$ is absolutely convergent in $\mathbb{K}$.*
2. *The series of functions $\sum_{n\in\mathbb{N}} u_n$ is uniformly convergent on I.*

Proof: Assume therefore $\sum_{n\in\mathbb{N}} \|u_n\|_\infty^I$ is convergent.
Since, for all $x \in I$, $|u_n(x)| \le \|u_n\|_\infty^I$, by comparison of series with positive terms, the series $\sum_{n\in\mathbb{N}} |u_n(x)|$ converges. This means that the series $\sum_{n\in\mathbb{N}} u_n(x)$ is absolutely convergent. It is therefore convergent, i.e. the series of functions $\sum_{n:\in\mathbb{N}} u_n$ converges pointwise on I. We will then have, for all $x \in I$

$$|R_n(x)| = \left|\sum_{k=n+1}^{+\infty} u_n(x)\right| \le \sum_{k=n+1}^{+\infty} |u_n(x)| \le \sum_{k=n+1}^{+\infty} \|u_n\|_\infty^I$$

hence $\|R_n\|_\infty^I \leq \underbrace{\sum_{k=n+1}^{+\infty} \|u_n\|^I}_{\text{remainder of aconvergent infinite series}} \underset{n\to+\infty}{\longrightarrow} 0$, which proves the uniform convergence of the series. □

Remark 2.2.21 *Using the abbreviations PWC, AC, UC, and NC for pointwise, absolute, uniform, and normal convergence respectively, we have the following sequence of implications:*

$$\sum_{n\in\mathbb{N}} u_n \ NC \text{ on } I \Longrightarrow \sum_{n\in\mathbb{N}} u_n(x) \ AC\ \forall x \in I \Longrightarrow \sum_{n\in\mathbb{N}} u_n \ PWC \text{ on } I$$

$$\sum_{n\in\mathbb{N}} u_n \ NC \text{ on } I \Longrightarrow \sum_{n\in\mathbb{N}} u_n \ UC \text{ on } I \Longrightarrow \sum_{n\in\mathbb{N}} u_n \ PWC \text{ on } I$$

Examples 2.2.22

1. *Study of the series of functions* $\displaystyle\sum_{n\geq 1} \frac{1}{x^2+n^2}$.

   Let, for all $x \in \mathbb{R}$, $u_n(x) = \dfrac{1}{x^2+n^2}$

   We have $|u_n(x)| \leq \dfrac{1}{n^2}$ *for all* x, *so* $\|u_n\|_\infty^{\mathbb{R}} \leq \dfrac{1}{n^2}$.

   The series with positive terms $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$ *being convergent, it follows from the comparison theorem for series with positive terms that the series* $\displaystyle\sum_{n\in\mathbb{N}} \|u_n\|_\infty^{\mathbb{R}}$ *converges.*

   Thus, the series $\sum_{n\in\mathbb{N}} u_n$ *is normally, therefore uniformly, convergent on* $\mathbb{R}$.

2. ***Important Example :*** *Study of the series of functions* $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}$.

   Let, for all $x \in \mathbb{R}$ *and all* $n \in \mathbb{N}^*$, $u_n(x) = \dfrac{(-1)^{n-1}x^n}{n}$.

   - Pointwise Convergence :
     1. *For* $|x| > 1$, $u_n(x)$ *does not tend to* 0 *as* $n \to +\infty$, *so the series* $\displaystyle\sum_{n\in\mathbb{N}^*} u_n(x)$ *diverges to infinity.*
     2. *For* $x = 1$ *the series converges (alternating harmonic series), and for* $x = -1$, *the series diverges (harmonic series).*
     3. *For* $|x| < 1$, *we have* $|u_n(x)| \leq |x|^n$. *Now the series with positive terms* $\displaystyle\sum_{n\in\mathbb{N}^*} |x|^n$ *is a geometric series with common ratio* $|x| < 1$, *therefore converges. The usual comparison theorems on series with positive real terms then ensure the absolute convergence, therefore the convergence, of the series* $\displaystyle\sum_{n\geq 1} u_n(x)$.

     In conclusion: the series converges pointwise on the interval $]-1,1]$.
   - Normal Convergence :

     There is no normal convergence on the whole interval $]-1,1]$. *Indeed,* $\|u_n\|_\infty = \dfrac{1}{n}$, *and the series* $\displaystyle\sum_{n\in\mathbb{N}^*} \frac{1}{n}$ *diverges!*

However, there is normal convergence (therefore uniform) on any interval of the form $[-a, a]$ *with* $0 \le a < 1$. *Indeed,* $\|u_n\|_\infty^{[-a,a]} = \frac{a^n}{n}$, *and the series* $\sum_{n\in\mathbb{N}^*} \frac{a^n}{n}$ *converges by d'Alembert's rule.*

- Uniform Convergence :

  There is no normal convergence on $[0, 1]$, *but let us show that there is uniform convergence on* $[0, 1]$.

  Indeed, by denoting $R_n(x) = \sum_{k=n+1}^{+\infty} u_k(x) = \sum_{k=n+1}^{+\infty} \frac{(-1)^{k-1}x^k}{k}$, $R_n(x)$ *is the remainder of order* n *of an alternating series which satisfies the assumptions of the alternating series test (immediate verification). We therefore have, for all* $x \in [0, 1]$, $|R_n(x)| \le \frac{x^{n+1}}{n+1} \le \frac{1}{n+1}$, *so* $\|R_n\|_\infty^{[0,1]} \le \frac{1}{n+1} \underset{n\to+\infty}{\longrightarrow} 0$, *which proves the uniform convergence on* $[0, 1]$ *(we easily deduce that there is then uniform convergence on any interval of the form* $[a, 1]$ *with* $-1 < a \le 0$*).*

2.2.9 Properties of the Sum of a Series of Functions

The following important theorem is assumed.

Theorem 2.2.23 (Interchanging Limits (or Double Limit)) *Let* $\sum_{n\in\mathbb{N}} u_n$ *be a series of functions defined on* I, *taking values in* $\mathbb{K}$.
Let $a \in \bar{I}$ *(possibly* $\pm\infty$*). Assume that, for any integer* n, *the limit* $\lim_{\substack{x\to a\\ x\in I}} u_n(x) = \ell_n$ *exists, and that the series* $\sum_{n\in\mathbb{N}} u_n$ *is* uniformly convergent in a neighborhood of a. *Let* $S = \sum_{n=0}^{+\infty} u_n$.
Then:

- *The series* $\sum_{n\in\mathbb{N}} \ell_n$ *converges*
- $\lim_{\substack{x\to a\\ x\in I}} S(x) = \sum_{n=0}^{+\infty} \ell_n$ *(that is, in short:* $\lim_a (\sum_{n=0}^{+\infty} u_n) = \sum_{n=0}^{+\infty} \lim_a u_n$*).*

The following theorems follow directly from the similar theorems concerning sequences of functions (we apply these theorems to the partial sums of the series of functions).

Theorem 2.2.24 (Continuity of the Sum) *Let* $\sum_{n\in\mathbb{N}} u_n$ *be a series of functions defined on an interval* I, *taking values in* $\mathbb{K}$, *such that the series* $\sum_{n\in\mathbb{N}} u_n$ *converges pointwise on* I. *Let* S *be its sum. Assume that:*

- *the* u_n *are continuous at* a*;*
- *there exists a neighborhood* V *of* a *such that* the series $\sum_{n\in\mathbb{N}} u_n$ converges uniformly on V.

Then S *is continuous at* a.

Corollary 2.2.25 *Let* $\sum_{n\in\mathbb{N}} u_n$ *be a series of functions defined on an interval* I, *taking values in* $\mathbb{K}$.
If the u_n *are continuous on* I *and if the series* converges uniformly locally on I, *then its sum* S *is continuous on* I.

Proof: Indeed, for all $a \in I$ there exists a neighborhood V of a such that the series $\sum u_n$ converges uniformly on V. By the previous theorem applied to V, S is continuous at a.
Thus S is continuous at every point of I, that is, on I. □

Theorem 2.2.26 (Interchanging Series-Integral on a Segment) *Let* $\sum_{n\in\mathbb{N}} u_n$ *be a series of functions defined on* a segment $[a,b] \subset \mathbb{R}$*, taking values in* $\mathbb{K}$.
Assume that the u_n are continuous on $[a,b]$*, and that* the series $\sum_{n\in\mathbb{N}} u_n$ converges uniformly on $[a,b]$*. Let* $S = \sum_{n=0}^{+\infty} u_n$.
Then S *is continuous on* $[a,b]$*, the series* $\sum_{n\in\mathbb{N}} \int_a^b u_n(t)dt$ *converges, and* $\int_a^b S(t)dt = \sum_{n=0}^{+\infty} \int_a^b u_n(t)dt$.

Proof: We prove the theorem in three steps.

Step 1: Continuity of the sum S

For each $n \in \mathbb{N}$, u_n is continuous on $[a,b]$ by hypothesis. Since the series $\sum_{n=0}^{\infty} u_n$ converges uniformly on $[a,b]$, the sum $S = \sum_{n=0}^{\infty} u_n$ is the uniform limit of the partial sums $S_N = \sum_{n=0}^{N} u_n$.

Each partial sum S_N is a finite sum of continuous functions, hence is continuous on $[a,b]$. The uniform limit of continuous functions on a compact interval is continuous. Therefore, S is continuous on $[a,b]$, and in particular is integrable on $[a,b]$.

Step 2: Convergence of the series of integrals

Let $\varepsilon > 0$. By uniform convergence of the series $\sum u_n$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and for all $x \in [a,b]$,

$$\left| \sum_{n=N+1}^{+\infty} u_n(x) \right| < \frac{\varepsilon}{b-a}.$$

Consider the partial sums of the series of integrals:

$$I_N = \sum_{n=0}^{N} \int_a^b u_n(t)\,dt.$$

For $M > N \geq N_0$, we have:

$$\begin{aligned}
|I_M - I_N| &= \left| \sum_{n=N+1}^{M} \int_a^b u_n(t)\,dt \right| \\
&\leq \sum_{n=N+1}^{M} \int_a^b |u_n(t)|\,dt \quad \text{(by triangle inequality)} \\
&\leq \int_a^b \left(\sum_{n=N+1}^{M} |u_n(t)| \right) dt \quad \text{(by linearity of the integral)} \\
&\leq \int_a^b \left| \sum_{n=N+1}^{M} u_n(t) \right| dt \quad \left(\text{since } \left|\sum a_n\right| \leq \sum |a_n|\right) \\
&< \int_a^b \frac{\varepsilon}{b-a}\,dt = \varepsilon.
\end{aligned}$$

This shows that (I_N) is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) is complete, the sequence (I_N) converges. Therefore, the series $\displaystyle\sum_{n=0}^{\infty} \int_a^b u_n(t)\,dt$ converges.

Step 3: Interchanging the integral and the series

Let $S(x) = \sum_{n=0}^{\infty} u_n(x)$ and $S_N(x) = \sum_{n=0}^{N} u_n(x)$. For any $N \in \mathbb{N}$, we have:

$$\int_a^b S_N(t)\,dt = \sum_{n=0}^{N} \int_a^b u_n(t)\,dt.$$

Now, since the convergence $S_N \to S$ is uniform on $[a,b]$, we have:

$$\sup_{x\in[a,b]} |S_N(x) - S(x)| \to 0 \quad \text{as } N \to \infty.$$

Therefore,

$$\begin{aligned}
\left| \int_a^b S(t)\,dt - \sum_{n=0}^{N} \int_a^b u_n(t)\,dt \right| &= \left| \int_a^b S(t)\,dt - \int_a^b S_N(t)\,dt \right| \\
&= \left| \int_a^b (S(t) - S_N(t))\,dt \right| \\
&\leq \int_a^b |S(t) - S_N(t)|\,dt \\
&\leq (b-a) \sup_{x\in[a,b]} |S(x) - S_N(x)|.
\end{aligned}$$

Taking the limit as $N \to \infty$, we obtain:

$$\lim_{N\to\infty} \left| \int_a^b S(t)\,dt - \sum_{n=0}^{N} \int_a^b u_n(t)\,dt \right| = 0.$$

This proves that:

$$\int_a^b S(t)\,dt = \lim_{N\to\infty} \sum_{n=0}^{N} \int_a^b u_n(t)\,dt = \sum_{n=0}^{\infty} \int_a^b u_n(t)\,dt.$$

Conclusion: We have shown that under the given hypotheses:

1. S is continuous on $[a,b]$,
2. The series $\sum_{n=0}^{\infty} \int_a^b u_n(t)\,dt$ converges,
3. $\displaystyle\int_a^b S(t)\,dt = \sum_{n=0}^{\infty} \int_a^b u_n(t)\,dt.$

This completes the proof. □

Remark 2.2.27 *The theorem also applies when the u_n are only* piecewise continuous*; however, it is necessary to check the piecewise continuity of S, which is no longer guaranteed by uniform convergence.*

Theorem 2.2.28 (Term-by-term Differentiation) *Let $\displaystyle\sum_{n\in\mathbb{N}} u_n$ be a series of functions defined on an interval I, taking values in $\mathbb{K}$.*
Assume that:

a) the u_n are of class C^1 on I;

b) the series of functions $\sum_{n\in\mathbb{N}} u_n$ converges pointwise on I; we will denote its sum by S;

c) the series of functions $\sum_{n\in\mathbb{N}} u'_n$ converges pointwise on I, the convergence is locally uniform *I.*

Then:

1. *The function S is of class C^1 on I;*
2. *the series of functions $\sum_{n\in\mathbb{N}} u_n$ converges uniformly locally on I;*
3. *for all $x \in I$, we have: $S'(x) = \sum_{n=0}^{+\infty} u'_n(x)$.*

Proof: We prove the theorem in several steps.

Step 1: Setup and notation

Let $S(x) = \sum_{n=0}^{\infty} u_n(x)$ be the pointwise sum (given by hypothesis (b)). Let $T(x) = \sum_{n=0}^{\infty} u'_n(x)$ be the pointwise sum of the derivatives (hypothesis (c) ensures this series converges pointwise).

Define the partial sums:

$$S_N(x) = \sum_{n=0}^{N} u_n(x) \quad \text{and} \quad T_N(x) = \sum_{n=0}^{N} u'_n(x).$$

Step 2: S_N converges uniformly on compact subsets of I

Let $K \subset I$ be a compact interval. We will show that (S_N) converges uniformly on K.

Fix $x_0 \in I$. For any $x \in K$ and $N \in \mathbb{N}$, by the fundamental theorem of calculus applied to each u_n (which is C^1), we have:

$$u_n(x) = u_n(x_0) + \int_{x_0}^{x} u'_n(t)\,dt.$$

Summing from $n = 0$ to N, we get:

$$S_N(x) = S_N(x_0) + \int_{x_0}^{x} T_N(t)\,dt.$$

Now consider $M > N \geq 0$. We have:

$$S_M(x) - S_N(x) = [S_M(x_0) - S_N(x_0)] + \int_{x_0}^{x} [T_M(t) - T_N(t)]\,dt.$$

By hypothesis (c), the series $\sum u'_n$ converges uniformly on K (since K is compact and convergence is locally uniform). Thus, (T_N) is uniformly Cauchy on K. That is, for any $\varepsilon > 0$, there exists N_0 such that for all $M > N \geq N_0$ and all $t \in K$:

$$|T_M(t) - T_N(t)| < \frac{\varepsilon}{2(1 + |K|)},$$

where $|K|$ denotes the length of the interval K.

Also, by hypothesis (b), $(S_N(x_0))$ is a Cauchy sequence in $\mathbb{K}$, so there exists N_1 such that for all $M > N \geq N_1$:

$$|S_M(x_0) - S_N(x_0)| < \frac{\varepsilon}{2}.$$

Taking $N_2 = \max(N_0, N_1)$, we have for all $M > N \geq N_2$ and all $x \in K$:

$$\begin{aligned}
|S_M(x) - S_N(x)| &\leq |S_M(x_0) - S_N(x_0)| + \left| \int_{x_0}^{x} [T_M(t) - T_N(t)]\, dt \right| \\
&\leq \frac{\varepsilon}{2} + \int_{\min(x_0,x)}^{\max(x_0,x)} |T_M(t) - T_N(t)|\, dt \\
&\leq \frac{\varepsilon}{2} + |x - x_0| \cdot \frac{\varepsilon}{2(1+|K|)} \\
&\leq \frac{\varepsilon}{2} + |K| \cdot \frac{\varepsilon}{2(1+|K|)} < \varepsilon.
\end{aligned}$$

This shows that (S_N) is uniformly Cauchy on K, hence converges uniformly on K. Since K was arbitrary, S_N converges to S uniformly on compact subsets of I.

Step 3: S is differentiable and $S' = T$

Fix $x_0 \in I$. Choose a compact interval $K \subset I$ containing x_0 in its interior. For any $x \in K$, we have for each N:

$$S_N(x) = S_N(x_0) + \int_{x_0}^{x} T_N(t)\, dt.$$

Taking the limit as $N \to \infty$, and using the uniform convergence established in Step 2:

- $S_N(x) \to S(x)$ and $S_N(x_0) \to S(x_0)$.
- Since $T_N \to T$ uniformly on K (by hypothesis (c)), and each T_N is continuous (as a finite sum of continuous functions u_n'), the limit T is continuous on K.
- By the theorem on interchanging limit and integral for uniform convergence (proved earlier), we have:
$$\lim_{N\to\infty} \int_{x_0}^{x} T_N(t)\, dt = \int_{x_0}^{x} \lim_{N\to\infty} T_N(t)\, dt = \int_{x_0}^{x} T(t)\, dt.$$

Therefore, passing to the limit in the equality, we obtain:

$$S(x) = S(x_0) + \int_{x_0}^{x} T(t)\, dt \quad \text{for all } x \in K.$$

Since T is continuous on K (as uniform limit of continuous functions), by the fundamental theorem of calculus, S is differentiable at x_0 and:

$$S'(x_0) = T(x_0) = \sum_{n=0}^{\infty} u_n'(x_0).$$

Step 4: S is C^1 on I

We have shown that $S' = T$ on I. Since each T_N is continuous (finite sum of continuous u_n') and $T_N \to T$ uniformly on compact subsets, T is continuous on every compact subset of I, hence continuous on I (since continuity is a local property).

Therefore, $S' = T$ is continuous on I, which means S is of class C^1 on I.

Step 5: Uniform local convergence of $\sum u_n$

This was already established in Step 2: for any compact $K \subset I$, the convergence $S_N \to S$ is uniform on K, which means $\sum u_n$ converges uniformly on compact subsets of I, i.e., converges locally uniformly on I.

Conclusion: We have proven all three claims:

1. S is of class C^1 on I,
2. $\sum u_n$ converges uniformly on compact subsets of I,
3. $S'(x) = \sum_{n=0}^{\infty} u'_n(x)$ for all $x \in I$.

This completes the proof of the theorem. $\square$

Corollary 2.2.29 (Series of Functions of Class C^k, $k \geq 1$) *Let $\sum_{n\in\mathbb{N}} u_n$ be a series of functions defined on an interval I, taking values in $\mathbb{K}$.*
Assume that:

a) the u_n are of class C^k on I;

b) each series of functions $\sum_{n\in\mathbb{N}} u_n^{(j)}$ for $j \in \{0, k-1\}$ converges pointwise on I;

c) the series of functions $\sum_{n\in\mathbb{N}} u_n^{(k)}$ converges pointwise on I, the convergence being local uniform on I.

Then: the sum function S is of class C^k on I, each series $\sum u_n^{(j)}$ with $j \in 0, k$ converges uniformly locally to $S^{(j)}$, and:

$$\forall j \in \{0, k\} \ , \ \forall x \in I \ , \ S^{(j)}(x) = \sum_{n=0}^{+\infty} u_n^{(j)}(x).$$

Corollary 2.2.30 (Series of Functions of Class C^∞) *Let $\sum_{n\in\mathbb{N}} u_n$ be a series of functions defined on an interval I, taking values in $\mathbb{K}$. Assume that:*

a) the u_n are of class C^∞ on I;

b) for all $j \in \mathbb{N}$, the series of functions $\sum_{n\in\mathbb{N}} u_n^{(j)}$ converges pointwise on I;

c) there exists an integer $p \in \mathbb{N}^*$ such that, for any integer $k \geq p$ the series of functions $\sum_{n\in\mathbb{N}} u_n^{(k)}$ converges pointwise on I, the convergence being local uniform on I.

Then: the sum function S is of class C^∞ on I, each series $\sum u_n^{(j)}$ with $j \in \mathbb{N}$ converges uniformly locally to $S^{(j)}$ and:

$$\forall j \in \mathbb{N} \ , \ \forall x \in I \ , \ S^{(j)}(x) = \sum_{n=0}^{+\infty} u_n^{(j)}(x).$$

2.3 Examples of Applications

Examples 2.3.1

1. Let $z \in \mathbb{C}$ and let

$$e_z = \begin{cases} \mathbb{R} & \longrightarrow \mathbb{C} \\ t & \longmapsto e^{tz} \end{cases}$$

Then the function e_z is of class C^∞ on $\mathbb{R}$ and: $\forall t \in \mathbb{R},\ e_z'(t) = z\mathrm{e}^{tz}$.

Let, for all $t \in \mathbb{R}$, $u_n(t) = \dfrac{t^n z^n}{n!}$, so that $e_z(t) = \displaystyle\sum_{n=0}^{+\infty} u_n(t)$. We have already seen that this series converges pointwise (absolutely) on $\mathbb{R}$ (cf. course on the complex exponential).

Furthermore, the u_n are of class $\mathbb{C}^1$ on $\mathbb{R}$ and $\forall t \in \mathbb{R}, \forall n \in \mathbb{N}^*$, $u_n'(t) = \dfrac{t^{n-1} z^n}{(n-1)!}$. The series $\sum_{n\geq 1} u_n'$ converges normally, therefore uniformly, on any segment of the form $[-A, A]$ with $A > 0$ since $\left\|u_n'\right\|_\infty^A = \dfrac{A^{n-1}\,|z|^n}{(n-1)!}$, general term of a convergent series (with sum $|z|\,\mathrm{e}^{A|z|}$).

The term-by-term differentiation theorem therefore applies: e_z is of class C^1 on $\mathbb{R}$ and for all real t, we have:

$$e_z'(t) = \sum_{n=1}^{+\infty} \frac{t^{n-1} z^n}{(n-1)!} = z \sum_{n=1}^{+\infty} \frac{t^{n-1} z^{n-1}}{(n-1)!} = z\mathrm{e}^{tz}.$$

Remark. *By induction we immediately have: $\forall n \in \mathbb{N}$, e_z is of class C^n on $\mathbb{R}$ and $\forall t \in \mathbb{R},\ e_z^{(n)}(t) = z^n \mathrm{e}^{tz}$. And in particular, for $z = \mathrm{i}$ we obtain*

$$e_{\mathrm{i}}^{(n)}(t) = \mathrm{i}^n \mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}\left(t + n\frac{\pi}{2}\right)}.$$

We thus find known results:

$$\forall n \in \mathbb{N},\ \forall t \in \mathbb{R},\ \cos^{(n)}(t) = \cos\left(t + n\frac{\pi}{2}\right) \qquad et \qquad \sin^{(n)}(t) = \sin\left(t + n\frac{\pi}{2}\right).$$

2. *Study of the series of functions $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n-1} x^n}{n}$.*

- Domain of definition :

  We have already seen that this series of functions converges pointwise on $]-1, 1]$; we can therefore define its sum

$$\forall x \in]-1, 1]\ ,\ S(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} x^n}{n}.$$

- Continuity :

  We have already seen that the series converges uniformly on any segment of the form $[a, 1]$ with $-1 < a \leq 0$. Since u_n is obviously continuous, it follows that S is continuous on any interval of this type, and thus on $(-1, 1)$.

- Differentiability :

  For all $n \in \mathbb{N}^*$, the function $u_n : x \mapsto \dfrac{(-1)^{n-1} x^n}{n}$ is of class C^1 on $-1, 1[$, and $u_n'(x) = (-1)^n x^{n-1}$.

  For all $a \in]0, 1[$, $\left\|u_n'\right\|_\infty^{[-a,a]} = a^{n-1}$, so the series $\displaystyle\sum_{n\geq 1} \left\|u_n'\right\|_\infty^{[-a,a]}$ is convergent (geometric series). It follows that the series $\sum_{n\geq 1} u_n'$ converges normally, therefore uniformly, on any segment included in $]-1, 1]$.

The term-by-term differentiation theorem then allows us to assert that S is of class C^1 on $]-1,1[$ and that

$$\forall x \in]-1,1[\ ,\ S'(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} x^{n-1} = \frac{1}{1+x}\,\cdot$$

S being continuous on $]-1,1[$, we deduce

$$\forall x \in]-1,1[\ ,\ S(x) = S(0) + \int_0^x S'(t)\,\mathrm{d}t = \ln(1+x)\,.$$

Finally, this last equality extends to $x=1$ by continuity. Therefore, we have proved : $\forall x \in]-1,1]\ ,\ \ln(1+x) = \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}x^n}{n}$.
(for $x=1$, We obtain the well-known value of the alternating harmonic series.).

3. *Study of the series of functions $\displaystyle\sum_{n\geq 1} \frac{a^n \cos nx}{n}$ with $0<a<1$.*

Let, for all $x\in\mathbb{R}$ and all $n\geq 1$: $u_n(x) = \dfrac{a^n\cos nx}{n}$.

- Domain of definition and continuity :
  For all $x\in\mathbb{R}$, we have $|u_n(x)| \leq \dfrac{a^n}{n} \leq a^n$, so $\|u_n\|_\infty^{\mathbb{R}} \leq a^n$. By comparison with a geometric series, we deduce that the series of functions $\sum_{n\geq 1} u_n$ is normally therefore uniformly convergent on $\mathbb{R}$.
  We can therefore set:

$$\forall x \in \mathbb{R}\ ,\ S(x) = \sum_{n=1}^{+\infty} \frac{a^n \cos nx}{n}\,.$$

  Moreover, since u_n is continuous on $\mathbb{R}$, S is also continuous by uniform convergence.

- Differentiability :
  For all $n\in\mathbb{N}^*$, the function u_n is of class C^1 on $\mathbb{R}$ and, for all $x\in\mathbb{R}$, $u_n'(x) = -a^n \sin nx$. We therefore have $\left\|u_n'\right\|_\infty^{\mathbb{R}} = a^n$, and the series of functions $\sum_{n\ geq1} u_n'$ converges normally therefore uniformly on $\mathbb{R}$.
  The term-by-term differentiation theorem then allows us to assert that S is of class C^1 on $\mathbb{R}$ and that

$$\forall x \in \mathbb{R}\ ,\ S'(x) = \sum_{n=1}^{+\infty} -a^n \sin nx = \sum_{n=0}^{+\infty} -a^n \sin nx\,.$$

  Let us calculate this sum:

$$\begin{aligned} S'(x) = -\Im\left(\sum_{n=0}^{+\infty} a^n \mathrm{e}^{\mathrm{i}nx}\right) &= -\Im\left(\sum_{n=0}^{+\infty} (a\mathrm{e}^{\mathrm{i}x})^n\right) \\ &= -\Im\left(\frac{1}{1-a\mathrm{e}^{\mathrm{i}x}}\right) = -\frac{a\sin x}{a^2-2a\cos x+1}. \end{aligned}$$

Now, for $x=0$, $S(0)=\sum_{n=1}^{+\infty}\frac{a^n}{n}=-\ln(1-a)$, *so we will have, for all* $x\in\mathbb{R}$*:*

$$\begin{aligned}S(x)=S(0)+\int_0^x S'(t)\,\mathrm{d}t &= -\ln(1-a)+\int_0^x -\frac{a\sin t}{a^2-2a\cos t+1}\,\mathrm{d}t\\ &= -\ln(1-a)-\frac{1}{2}\big[\ln\left|a^2-2a\cos t+1\right|\big]_0^x\\ &= -\ln(1-a)+\frac{1}{2}\ln((1-a)^2-\frac{1}{2}\ln\left|a^2-2a\cos x+1\right|\\ &= -\frac{1}{2}\ln(a^2-2a\cos x+1)\quad \textit{(since } a\in]0,1[)\end{aligned}$$

- Application : *Calculate* $I=\int_0^{\pi}\ln(a^2-2a\cos x+1)\,\mathrm{d}x$.

  Thus, $I=-2\int_0^{\pi}S(x)\,\mathrm{d}x$.

  The series $S(x)=\sum_{n=1}^{+\infty}\frac{a^n\cos nx}{n}$ *being normally therefore uniformly convergent on* $\mathbb{R}$, *therefore on* $[0,\pi]$, *we can apply the term-by-term integration theorem:*

$$\int_0^{\pi}S(x)\,\mathrm{d}x=\sum_{n=1}^{+\infty}\int_0^{\pi}\frac{a^n\cos nx}{n}\,\mathrm{d}x=0$$

  hence: $I=0$.

2.4 Exercises of the Chapter

Exercise 2.4.1 *Let* $(f_n), f_n:[a,b]\to\mathbb{R}$ *denote a sequence of functions. Define what it means for* (f_n) *to converge pointwise on* $[a,b]$.

(i) Determine the pointwise limit function in the case

$$f_n:(-1,1]\to\mathbb{R},\quad f_n(x):=x^n,\quad n=1,2,\cdots$$

Explain why the convergence is not uniform.

(ii) Show that the pointwise limit function of the sequence

$$f_n:[0,1]\to\mathbb{R},\quad f_n(x):=\frac{nx}{1+n^2x^2},\quad n=1,2,\cdots$$

is the zero function but that

$$\max\{|f_n(x)|:0\le x\le 1\}\nrightarrow 0\qquad \text{as } n\to\infty.$$

Correction 2.4.1 (f_n) *converges pointwise on* $[a,b]$ *if the sequence of numbers* $(f_n(x))$ *converges for all* $x\in[a,b]$.

(i) If $|x|<1$ *then* $x^n\to 0$ *as* $n\to\infty$. *If* $x=1$ *then* $x^n=1$ *for all* n. *The pointwise limit function is* $f:(-1,1]\to\mathbb{R}$,

$$f(x) := \begin{cases} 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

Each function f_n is continuous on $(-1,1]$ and the limit function is f is discontinuous at $x = 1$ which implies that the convergence is not uniform.

(ii) *For $x = 0$ we have $f_n(0) = 0$.*

For $x > 0$ we have

$$f_n(x) = \frac{x/n}{1/(n^2) + x^2} \to \frac{0}{0 + x^2} = 0 \qquad \text{as } n \to \infty.$$

Thus for all $x \in [0,1]$ we have $f_n(x) \to 0$ as $n \to \infty$.
By inspection when $x = 1/n$ we have $f_n(1/n) = 1/(1+1) = 1/2$ for all n. Thus

$$\max\{|f_n(x)| : 0 \leq x \leq 1\} \geq f_n(1/n) = \frac{1}{2}$$

and we hence do not have convergence to 0.

Exercise 2.4.2

(a) *Let $I \subset \mathbb{R}$ and let f be a bounded function on I. Define the uniform norm $\|f\|$ of f on I.*

(b) *Let $I \subset \mathbb{R}$ and let $f_n : I \to \mathbb{R}, n = 1, 2, \cdots$ and $f : I \to \mathbb{R}$ be functions.*

(i) *Define what it means for (f_n) to converge pointwise on I.*

(ii) *Define what it means for (f_n) to converge to f uniformly on I.*

(c) *In each of the following cases of sequences of functions, determine the pointwise limit function and determine whether or not the convergence is uniform.*

(i)

$$f_n : [0,\infty) \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{1 + x^n}.$$

(ii)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \frac{\cos(nx)}{n}.$$

(iii)

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) := x^n(1 - x).$$

(iv)

$$f_n : [0,\infty) \to \mathbb{R}, \quad f_n(x) := x\mathrm{e}^{-nx}.$$

Correction 2.4.2

(a)

$$\|f\| := \sup\{|f(x)| : x \in I\}.$$

(b) *(i)* *(f_n) converges pointwise on I if the sequence of numbers $(f_n(x))$ converges for every $x \in I$.*

(ii) *(f_n) converges uniformly on I to $f, f : I \to \mathbb{R}$, if*

$$\|f_n - f\| \to 0 \qquad \text{as } n \to \infty$$

(c) (i) *If* $0 \le x < 1$ *then* $x^n \to 0$ *as* $n \to \infty$ *and we get* $f_n(x) \to 0$ *as* $n \to \infty$.
If $x = 1$ *then* $f_n(1) = 1/2 \to 1/2$ *as* $n \to \infty$.
If $x > 1$ *then*

$$f_n(x) = \frac{1}{(1/x^n) + 1} \to \frac{1}{0+1} = 1 \qquad \text{as } n \to \infty$$

Hence the pointwise limit function f *is the discontinuous function*

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) := \begin{cases} 0, & 0 \le x < 1 \\ 1/2, & x = 1 \\ 1, & x > 1 \end{cases}$$

As each f_n *is continuous and the pointwise limit is discontinuous the convergence is not uniform.*

(ii) *For all* $x \in \mathbb{R}$

$$\left| \frac{\cos(nx)}{n} \right| \le \frac{1}{n} \to 0 \qquad \text{as } n \to \infty$$

Thus (f_n) *converges uniformly to the zero function on* $\mathbb{R}$ *which is thus the pointwise limit.*

(iii) *If* $x = 1$ *then* $f_n(1) = 0$ *and* $(f_n(1))$ *is a constant sequence. If* $0 \le x < 1$ *then* $x^n \to 0$ *as* $n \to \infty$. *Thus* (f_n) *converges pointwise to the zero function on* $[0, 1]$. *To test for uniform convergence we determine the uniform norm of each* f_n.
As $f_n(x) \ge 0$ *with* $f_n(0) = f_n(1) = 0$ *we need to find the maximum of* $f_n(x)$, $0 < x < 1$.

$$f_n'(x) = nx^{n-1} - (n+1)x^n = x^{n-1}(n - (n+1)x)$$

The maximum occurs at $x = n/(n+1)$. *Since for this* $x, x^n < 1$ *we have*

$$|f_n(n/(n+1))| \le 1 - \frac{n}{n+1} = \frac{1}{n+1} \to 0 \qquad \text{as } n \to \infty$$

Hence $f_n \to 0$ *uniformly on* $[0, 1]$.

(iv) *If* $x = 0$ *then* $f_n(0) = 0$ *and* $(f_n(0))$ *is a constant sequence. For* $x > 0$ $f_n(x) \to 0$ *as* $n \to \infty$. *Thus* (f_n) *converges pointwise to the zero function on* $[0, 1]$.
To test for uniform convergence we determine the uniform norm of each f_n.

$$f_n'(x) = \mathrm{e}^{-nx}(1 - nx) = 0 \qquad \text{when } x = \frac{1}{n}$$

As $f_n'(x) > 0$ *in* $[0, 1/n)$ *and* $f_n'(x) < 0$ *in* $(1/n, \infty)$ *this is a local maximum. We have*

$$\|f_n\| = f_n(1/n) = \frac{\mathrm{e}^{-1}}{n} \to 0 \qquad \text{as } n \to \infty$$

The sequence converges uniformly to the zero function.

Exercise 2.4.3

(a) Show that when $I = [0,1]$ and

$$f_n(x) := \frac{nx}{1+n^2x^2},$$

$\|f_n\| = 1/2$ for all n. Hence show that the sequence (f_n) converges pointwise on I but not uniformly on I.

(b) Suppose that (f_n) is a sequence of continuous functions which converges pointwise on I to f. What can you conclude about the uniformity of the convergence in the following cases: (1) when f is continuous on I, (2) when f is discontinuous on I.

(c) Let $f_n : I \to \mathbb{R}, n = 1, 2, \cdots$ be a sequence of bounded functions. Define what it means for (f_n) to be a Cauchy sequence in the uniform norm and show that if $f_n \to f$ uniformly then (f_n) is a Cauchy sequence. (In your answer you can assume that the uniform norm satisfies all the norm axioms, e.g. the triangle inequality.)

Correction 2.4.3

(a) As f_n is continuous on $[0,1]$ it attains its maximum on $[0,1]$.$f_n(0) = 0$ and $f_n(1) = n/\left(1+n^2\right) = 1/((1/n)+n)$.$f_1(1) = 1/2$ and for $n \geq 2 f_n(1) < 1/n \leq 1/2$. To find the maximum we consider turning points of f_n.

$$f_n'(x) = \frac{\left(1+n^2x^2\right)n - (nx)\left(2n^2x\right)}{\left(1+n^2x^2\right)^2} = \frac{n\left(1-n^2x^2\right)}{\left(1+n^2x^2\right)^2} = 0$$

when $x = 1/n$.f_n increases in $[0, 1/n)$ and decreases in $(1/n, \infty)$. Thus

$$\|f_n\| = f_n(1/n) = \frac{1}{2}$$

For the pointwise convergence observe that for $x = 0, f_n(x) = 0$ for all n. For $x > 0$,

$$f_n(x) = \frac{1/(nx)}{(1/nx)^2+1} \to \frac{0}{0+1} = 0 \qquad \text{as } n \to \infty$$

Thus the sequence converges pointwise to the zero function. As $\|f_n\| \nrightarrow 0$ the sequence does not converge uniformly.

(b) If the limit function is discontinuous then this is sufficient to prove that the convergence is not uniform. If the limit function is continuous then nothing can be concluded about whether or not the convergence is uniform.

(c) (f_n) is a Cauchy sequence if for every $\varepsilon > 0$ there exists an N such that

$$\|f_n - f_m\| < \varepsilon, \qquad \text{for all } n \geq N \text{ and } m \geq N$$

(f_n) converges uniformly to f means that for every $\varepsilon > 0$ there exists a N such that

$$\|f_n - f\| < \varepsilon/2 \qquad \text{for all } n \geq N$$

Then for all $m, n \geq N$ and for all $x \in I$ we compare both f_m and f_n with the limit f to give

$$\begin{aligned}|f_m(x) - f_n(x)| &= |(f_m(x) - f(x)) + (f(x) - f_n(x))| \\ &\le |f_m(x) - f(x)| + |f(x) - f_n(x)| \\ &\le \|f_m - f\| + \|f - f_n\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

As this is true for for all $x \in I$ we have $\|f_m - f_n\| < \varepsilon$ as required.

Exercise 2.4.4 *For $n = 1, 2, 3, \ldots$, x real, put*

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to +\infty} f_n^{'}(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Correction 2.4.4 *The Schwarz inequality, which implies that $|f_n(x)| \le \dfrac{|x|}{2\sqrt{n}|x|} = \dfrac{1}{2\sqrt{n}}$ for $x \neq 0$, shows that $f_n(x)$ tends uniformly to 0. Now $f_n^{'}(x) = \dfrac{1 - nx^2}{(1 + nx^2)^2}$, which tends to 0 if $x \neq 0$, though $f_n^{'}(0) = 1$ for all n.*

Exercise 2.4.5

(a) Let $(f_n), f_n : I \to \mathbb{R}$, denote a sequence of continuous functions defined on I. If (f_n) converges to f uniformly on I then what properties will the limit function f have?

(b) In each of the following cases of sequences of functions, determine whether or not the sequence converges pointwise on its given domain. If the sequence does converge pointwise then give the pointwise limit function and determine whether or not the convergence is uniform.

(i)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \frac{\sin nx}{n} \quad n = 1, 2, \cdots .$$

(ii)

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \cos nx.$$

(iii)

$$f_n : (-1, 1] \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{(1 + x^n)^2}, \quad n = 1, 2, \cdots$$

(iv)

$$f_n : [0, \infty) \to \mathbb{R}, \quad f_n(x) := \frac{x^n}{(1 + x^n)^n}, \quad n = 1, 2, \cdots$$

Correction 2.4.5

(a) As each f_n is continuous and (f_n) converges to f uniformly on I then f is continuous on I.

(b) *(i)* *By the properties of the sine function we have for all* $x \in \mathbb{R}$ *that*

$$|f_n(x)| \leq \frac{1}{n}$$

and thus

$$\|f_n\| \leq \frac{1}{n} \to 0 \qquad \text{as } n \to \infty$$

Thus the sequence converges uniformly to the function f

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) := 0$$

Uniform convergence implies pointwise convergence and hence f *is also the pointwise limit.*

(ii) *If we let* $x = \pi$ *then* $f_n(\pi) = \cos n\pi = (-1)^n$. *The sequence of numbers* $(f_n(\pi))$ *does not converge and thus the sequence does not converge pointwise and as a consequence it does not converge uniformly.*

(iii)

$$x^n \to 0 \qquad \text{as } n \to \infty \text{ if } |x| < 1$$

Thus for $x \in (-1, 1)$, $f_n(x) \to 0/(1 + 0)^2 = 0$ *as* $n \to \infty$. *Also,* $f_n(1) = 1/4$ *for all* n. *Thus*

$$f(x) := \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x \in (-1, 1) \\ 1/4, & \text{if } x = 1 \end{cases}$$

As each f_n *is continuous and* f *is discontinuous this indicates that the convergence is not uniform.*

(iv) *Let* $g_n : \mathbb{R} \to \mathbb{R}$,

$$g_n(x) := \frac{x}{1 + x^n} = \frac{(1/x)^{n-1}}{(1/x)^n + 1}.$$

If $0 \leq x < 1$ *then*

$$f_n(x) = \left(\frac{x}{1 + x^n}\right)^n \leq x^n \to 0 \qquad \text{as } n \to \infty.$$

If $x = 1$ *then*

$$f_n(1) = (1/2)^n \to 0 \qquad \text{as } n \to \infty$$

If $x > 1$ *then* $(1/x)^n \to 0$ *as* $n \to \infty$ *and*

$$f_n(x) = \left(\frac{(1/x)^{n-1}}{(1/x)^n + 1}\right)^n < \left(\frac{1}{x}\right)^{n(n-1)} \to 0 \qquad \text{as } n \to \infty$$

The pointwise limit function is $f : \mathbb{R} \to \mathbb{R}, f(x) := 0$.
To establish that the convergence is uniform we need to bound g_n *and hence* f_n. *We note that* $g_n(0) = 0$ *and* $g_n(x) \to 0$ *as* $|x| \to \infty$. *For turning values we have*

$$g_n'(x) = \frac{(1 + x^n) - x\left(nx^{n-1}\right)}{(1 + x^n)^2} = 0 \qquad \text{when} \quad 1 = (n - 1)x^n$$

As there is only one turning value it is the point where g_n has a global maximum. Thus

$$\max_{\mathbb{R}} |g_n(x)| \leq \frac{(1/(n-1))^{1/n}}{1+1/(n-1)} < \left(\frac{1}{(n-1)}\right)^{1/n}$$

and

$$\|f_n\| = \max_{\mathbb{R}} |f_n(x)| \leq \frac{1}{n-1} \to 0 \qquad \text{as } n \to \infty$$

Exercise 2.4.6

1. *Let the sequence of functions $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ on the interval $\left[0, \frac{\pi}{2}\right]$. Show that the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly to a differentiable function f and verify that the sequence $(f_n')_{n\in\mathbb{N}^*}$ does not converge.*

2. *Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$. Show that each f_n is of class C^1 and that the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges uniformly on $\mathbb{R}$ to a function f which is not of class C^1.*

Correction 2.4.6

1. *For all $x \in \left[0, \frac{\pi}{2}\right]$:*

$$\lim_{n\to+\infty} \frac{\sin(nx)}{\sqrt{n}} = 0$$

The sequence of functions (f_n) converges pointwise to zero function on $\left[0, \frac{\pi}{2}\right]$, and this function is obviously differentiable.

$$f_n'(x) = \frac{n\cos(nx)}{\sqrt{n}} = \sqrt{n}\cos(nx).$$

Except for $x = 0$, the sequence (f_n') does not have a limit.

2.

$$\lim_{n\to+\infty} \sqrt{x^2 + \frac{1}{n^2}} = |x|$$

The simple limit is $|x|$.

Next, we show that there is uniform convergence:

$$\left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| = \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| = \frac{\left(\sqrt{x^2+\frac{1}{n^2}} - |x|\right)\left(\sqrt{x^2+\frac{1}{n^2}} + |x|\right)}{\sqrt{x^2+\frac{1}{n^2}} + |x|} = \frac{x^2+\frac{1}{n^2}-x^2}{\sqrt{x^2+\frac{1}{n^2}} + |x|}$$

$$= \frac{\frac{1}{n^2}}{\sqrt{x^2+\frac{1}{n^2}} + |x|} = \frac{1}{n^2}\cdot\frac{1}{\sqrt{x^2+\frac{1}{n^2}} + |x|} \leq \frac{1}{n^2}\cdot\frac{1}{\sqrt{0+\frac{1}{n^2}} + 0} = \frac{1}{n^2}\cdot n = \frac{1}{n}$$

Therefore

$$\sup_{x\in\mathbb{R}} \left|\sqrt{x^2+\frac{1}{n^2}} - |x|\right| \leq \frac{1}{n}$$

And finally

$$\lim_{n\to+\infty}\sup_{x\in\mathbb{R}}\left|\sqrt{x^2+\frac{1}{n^2}}-|x|\right|=0$$

The sequence of functions (f_n) converges uniformly to $f(x)=|x|$, a function that is not differentiable at 0, and therefore is not of class C^1 on $\mathbb{R}$.

Exercise 2.4.7 *Let $f_n:[0,1]\to\mathbb{R}$ be defined by:*

$$f_n(x)=\begin{cases} n^2x(1-nx) & \text{for } x\in\left[0,\frac{1}{n}\right] \\ 0 & \text{otherwise}\end{cases}$$

1. *Study the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$.*
2. *Compute:*

$$\int_0^1 f_n(t)\,dt$$

   Is there uniform convergence of the sequence of functions $(f_n)_{n\in\mathbb{N}}$?

3. *Study the uniform convergence on $[a,1]$ with $a>0$.*

Correction 2.4.7

1. For $x\in]0,1]$, there exists n_0 such that $\frac{1}{n_0}<x$, so for all $n\geq n_0$, $f_n(x)=0\to 0$.

   For $x=0$, we have $f_n(0)=0$.

   Therefore, the sequence of functions (f_n) simply converges to 0.

2.

$$\int_0^1 f_n(t)dt=\int_0^{\frac{1}{n}} n^2t(1-nt)dt=n^2\int_0^{\frac{1}{n}}(t-nt^2)dt=n^2\left[\frac{t^2}{2}-\frac{nt^3}{3}\right]_0^{\frac{1}{n}}=n^2\left(\frac{1}{2n^2}-\frac{n}{3n^3}\right)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

   If there were uniform convergence of the sequence of functions (f_n), we would have

$$\lim_{n\to+\infty}\int_0^1 f_n(t)dt=\int_0^1 0dt=0$$

   Which is not the case, so there is no uniform convergence of the sequence of functions (f_n) towards zero function.

3. On $[0,1]$, the sequence of functions (f_n) converges pointwise to f, for all $n>\frac{1}{a}$, and for all $x\in[0,1]$, $f_n(x)=0$ thus

$$f_n(x)-f(x),[0,1]=0$$

   Therefore, there is uniform convergence.

Exercise 2.4.8 *The Weierstrass M-test gives a sufficient condition for the series $\sum f_n$ to converge uniformly on $[a,b]$. State the conditions of this test.*
Use the test to show that the following series converge uniformly on the given domain.

(i)

$$\sum_0^\infty \frac{x^n}{n^n} \qquad on\ |x| \le r < \infty.$$

(ii)

$$\sum_0^\infty \frac{(n!)^2}{(2n)!}x^n \qquad on\ |x| \le r < 4.$$

(iii)

$$\sum_0^\infty \frac{1}{x^2+2^n} \qquad on\ \mathbb{R}.$$

Correction 2.4.8 *The conditions of the Weierstrass M-test are satisfied if* $\|f_n\|_\infty \le M_n$ *with the series* $\sum M_n$ *converging.*

(i) In this case $f_n(x) = x^n/n^n$. *On the domain* $\{x : |x| \le r\}$ *we have*

$$\|f_n\|_\infty = \frac{r^n}{n^n} =: M_n$$

We test for the convergence of the series $\sum M_n$ *by using the root test.*

$$M_n^{1/n} = \frac{r}{n} \to 0 \qquad as\ n \to \infty$$

As the limit is less than 1 the series converges and by the M-test the power series converges uniformly.

(ii) In this case $f_n(x) = (n!)^2 x^n/(2n)\ !$. *On the domain* $\{x : |x| \le r\}$ *we have*

$$\|f_n\|_\infty = \frac{(n!)^2}{(2n)!}r^n =: M_n$$

We test for the convergence of the series $\sum M_n$ *by using the ratio test.*

$$\begin{aligned}\frac{M_{n+1}}{M_n} &= \frac{(n+1)^2}{(2n+2)(2n+1)}r \\ &= \frac{(1+1/n)^2}{(2+2/n)(2+1/n)}r \to \frac{r}{4} \qquad as\ n \to \infty\end{aligned}$$

As the limit is less than 1 when $r < 4$ *the series converges and by the M-test the power series converges uniformly.*

(iii) In this case $f_n(x) = 1/\left(x^2 + 2^n\right)$ *and on the domain* $\mathbb{R}$ *we have by inspection that* $\|f_n\|_\infty = f_n(0) = 1/2^n$. *The geometric series* $\sum_{n=0}^\infty 1/2^n$ *is convergent and by the M-test the series converges uniformly on* $\mathbb{R}$.

Exercise 2.4.9

(a) *We know from the previous chapter that the series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for all $p > 1$ *and diverges for all* $p \leq 1$.
Let

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) := \sum_{k=1}^{n} \frac{\cos(kx)}{k^3}$$

Use the the Weierstrass M-test to explain why (f_n) *and* (f_n') *converge uniformly on* $\mathbb{R}$.

Does the sequence (f_n'') *converge pointwise on* $\mathbb{R}$ *?*

(b) *Determine the radius of convergence of the following power series and state regions in which the series converge uniformly.*

(i)

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(ii) *In the following* $\alpha \in \mathbb{R}$ *is not an integer.*

$$\sum_{k=0}^{\infty} \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \cdots$$

(iii)

$$\sum_{k=0}^{\infty} \frac{k^3 x^k}{3^k}.$$

(iv)

$$\sum_{k=0}^{\infty} k^k x^k.$$

Correction 2.4.9

(a) *To apply the M-test to the series for* $f_n(x)$ *we note that* $|\cos(kx)| \leq 1$ *so that the* k *th component function is bounded by*

$$M_k = \frac{1}{k^3}.$$

The series $\sum 1/k^3$ *converges and thus the series for* f_n *converges uniformly on* $\mathbb{R}$. f_n' *is given by*

$$f_n'(x) = -\sum_{k=1}^{n} \frac{\sin(kx)}{k^2}$$

The k *th component function is bounded by*

$$M_k = \frac{1}{k^2}$$

The series $\sum 1/k^2$ *converges and thus the series for* f_n' *converges uniformly on* $\mathbb{R}$. f_n'' *is given by*

$$f_n''(x) = -\sum_{k=1}^{n} \frac{\cos(kx)}{k}$$

When $x = 0, \cos(0) = 1$ *and* $-f_n''(0)$ *is the nth partial sum of the divergent harmonic series. Hence the series does not converge pointwise on* $\mathbb{R}$.

(b) *(i) Let* $f_k(x) = x^k/k\,!$. *On* $|x| \le r$

$$|f_k(x)| \le \frac{r^k}{k!} =: M_k$$

By the ratio test

$$\frac{M_{k+1}}{M_k} = \frac{r^{k+1}/(k+1)!}{r^k/k!} = \frac{r}{k+1} \to 0 \qquad \text{as } k \to \infty$$

The series $\sum M_k$ *converges for all* r *and hence the radius of convergence is* ∞. *The series converges uniformly in any region of the form* $|x| \le r$.

(ii) Let $f_k(x) = a_k x^k$ *where*

$$a_k := \left(\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\right).$$

On $|x| \le r$

$$|f_k(x)| \le |a_k|\, r^k =: M_k$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \left|\frac{a_{k+1}}{a_k}\right| r = \left|\frac{\alpha-k}{k+1}\right| r = \left|\frac{\alpha/k-1}{1+1/k}\right| r \to r \qquad \text{as } k \to \infty$$

Thus the series converges absolutely if $r < 1$ *and diverges for* $r > 1$. *The radius of convergence is* $R = 1$ *and the series converges uniformly in* $[-r, r]$ *for all* r *satisfying* $0 \le r < 1$.

(iii) Let $f_k(x) = a_k x^k$ *where*

$$a_k := \frac{k^3}{3^k}$$

On $|x| \le r$

$$|f_k(x)| \le |a_k|\, r^k =: M_k$$

Using the ratio test

$$\frac{M_{k+1}}{M_k} = \frac{a_{k+1}}{a_k} r = \frac{(k+1)^3}{3k^3} r = \frac{(1+1/k)^3}{3} r \to \frac{r}{3}.$$

The series converges if $r < 3$ *and diverges for* $r > 3$. *The radius of convergence is* $R = 3$. *The series converges uniformly in* $|x| \le r$ *for all* $r < 3$.

(iv) *Let* $f_k(x) = k^k x^k$. *On* $|x| \leq r$

$$|f_k(x)| \leq (kr)^k =: M_k$$

Using the root test

$$M_k^{1/k} = kr.$$

The sequence $\left(M_k^{1/k}\right)$ *only converges when* $r = 0$ *The radius of convergence is* $R = 0$.

Exercise 2.4.10

(a) *Show that if* $f_n(x) := x + 1/n$ *and* $f(x) := x$ *for all* $x \in \mathbb{R}$ *then* $f_n \to f$ *uniformly on* $\mathbb{R}$ *but that* $\left(f_n^2\right)$ *does not converge uniformly on* $\mathbb{R}$.

(b) *Let* $C^{(1)}[a, b]$ *denote the set of continuously differentiable functions on a finite interval* $[a, b]$. *For each* f *in* $C^{(1)}[a, b]$, *define*

$$\|f\|_{C^1} := \|f\| + \|f'\|$$

Show that $\|\cdot\|_{C^1}$ *satisfies the norm requirements of non-negativity, linearity and the triangle inequality. (Remark: It can be shown that the linear space* $C^{(1)}[a, b]$ *is complete in this norm.)*

(c) *Use the Weierstrass M-test to deduce that if* $\sum |a_n|$ *and* $\sum |b_n|$ *converge then the Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges uniformly on $\mathbb{R}$.

(d) *Let*

$$s_n(x) := \sum_{k=1}^{n} \frac{\cos(kx)}{k^5}$$

Explain why $(s_n), (s_n'), (s_n'')$ *and* (s_n''') *all converge uniformly on* $\mathbb{R}$ *but that* $(s_n''''(0))$ *does not converge.*

Correction 2.4.10

(a) $f_n(x) = x + 1/n$ *and* $f(x) = x$ *and hence* $f_n(x) - f(x) = 1/n$. *Thus*

$$\|f_n - f\|_\infty = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$
$$f_n^2 - f^2 = (f_n + f)(f_n - f)$$

and thus in this case

$$f_n(x)^2 - f(x)^2 = \frac{1}{n}\left(2x + \frac{1}{n}\right)$$

and

$$\sup_{x\in\mathbb{R}} \left|f_n(x)^2 - f(x)^2\right| = \frac{1}{n} \sup_{x\in\mathbb{R}} \left|2x + \frac{1}{n}\right| = \infty \quad \text{for all } n.$$

(b) *Nonnegativity: Clearly* $\|f\|_{C^1} \geq 0$ *since* $\|f\| \geq 0$ *and* $\|f'\| \geq 0$. *If* $\|f\|_{C^1} = 0$ *then* $\|f\| = 0$ *and hence* $f(x) = 0$ *for all* $x \in [a,b]$.
Linearity: By using the linearity of the uniform norm

$$\|\alpha f\|_{C^1} = \|\alpha f\| + \|\alpha f'\| = |\alpha|\|f\| + |\alpha|\,\|f'\| = |\alpha|\|f\|_{C^1}$$

Triangle inequality: By the triangle inequality for the uniform norm

$$\begin{aligned}\|f+g\|_{C^1} &= \|f+g\| + \|f'+g'\| \\ &\leq (\|f\| + \|g\|) + (\|f'\| + \|g'\|) \\ &= \|f\|_{C^1} + \|g\|_{C^1}\end{aligned}$$

(c) *With* $f_n(x) := a_n \cos nx + b_n \sin nx$ *we have*

$$|f_n(x)| \leq |a_n| + |b_n|$$

by the triangle inequality and that $|\cos nx| \leq 1$ *and* $|\sin nx| \leq 1$. *As* $\sum |a_n|$ *and* $\sum |b_n|$ *both converge we have that* $\sum (|a_n| + |b_n|)$ *also converges. Hence the conditions of the Weierstrass* M*-test apply with* $M_n = |a_n| + |b_n|$ *and the series converges uniformly on* $\mathbb{R}$.

(d)

$$\begin{aligned}s_n(x) &:= \sum_{k=1}^{n} \frac{\cos(kx)}{k^5} \\ s_n'(x) &:= -\sum_{k=1}^{n} \frac{\sin(kx)}{k^4} \\ s_n''(x) &:= -\sum_{k=1}^{n} \frac{\cos(kx)}{k^3} \\ s_n'''(x) &:= \sum_{k=1}^{n} \frac{\sin(kx)}{k^2} \\ s_n''''(x) &:= \sum_{k=1}^{n} \frac{\cos(kx)}{k}\end{aligned}$$

The k *th term in the series of* s_n, s_n', s_n'' *and* s_n''' *are bounded on* $\mathbb{R}$ *by respectively* $1/k^5, 1/k^4, 1/k^3$ *and* $1/k^2$ *and as the series of the bounds converge the sequence of functions converge uniformly on* $\mathbb{R}$. *However*

$$s_n''''(0) = \sum_{k=1}^{n} \frac{1}{k}$$

are the partial sums of the harmonic series which is divergent.

Exercise 2.4.11 *We consider the function* $f : x \mapsto \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+x}$.

a) *Study the definition, continuity, and differentiability of* f *on* $\mathbb{R}_+^*$ *(the set of positive real numbers).*

b) *Show that* f *is decreasing on* $\mathbb{R}_+^*$.

c) *Find a relationship between $f(x)$ and $f(x+1)$.*

d) *Compute $f(p)$ for $p \in \mathbb{N}$ (natural numbers).*

e) *Determine an asymptotic equivalent of $f(x)$ near 0 and $+\infty$.*

f) *Show that: $\forall x > 0,\ f(x) = \displaystyle\int_0^1 \frac{t^{x-1}}{1+t}\,dt$.*

Correction 2.4.11

a) *The functions $f_n : x \mapsto \frac{(-1)^n}{n+x}$ are of class C^1 and $f_n'(x) = \dfrac{(-1)^{n+1}}{(n+x)^2}$.*

By the special criterion for alternating series, $\sum_{n\geq 0} f_n(x)$ converges pointwise on $\mathbb{R}_{\geq 0}$ to f.

Let $a > 0$. On $[a,+\infty[$, $\|f_n'\|_\infty^{[a,+\infty[} \leq \frac{1}{(n+a)^2}$ and the series $\sum_{n\geq 0} \frac{1}{(n+a)^2}$ converges, therefore $\sum f_n'$ converges normally and thus uniformly on $[a,+\infty[$.

According to the theorem on the differentiation of a series of functions, f is defined and of class C^1 on $\mathbb{R}_{\geq 0}$ and

$$\forall x > 0,\ f'(x) = \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(n+x)^2}.$$

b) *We can apply the special criterion for alternating series to the series $\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(n+x)^2}$ (easy verification). This series is thus of the sign of its first term $\frac{-1}{x^2}$. Therefore, $f'(x) \leq 0$ and the function f is decreasing.*

c)

$$f(x+1) + f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+x+1} + \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+x} = -\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x} + \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+x} = \frac{1}{x}.$$

d) *We use the previous relation to write, for integer $p \geq 2$:*

$$\begin{aligned} f(p) + (-1)^p f(1) &= \big(f(p) + f(p-1)\big) - \big(f(p-1) + f(p-2)\big) + \cdots + (-1)^p\big(f(2) + f(1)\big) \\ &= \frac{1}{p-1} - \frac{1}{p-2} + \cdots + (-1)^p. \end{aligned}$$

Since $f(1) = \ln 2$, we obtain $f(p)$.

e)

- *$f(x) = \frac{1}{x} - f(x+1)$ and $f(x+1) \to f(1)$ as $x \to 0$ by continuity, hence $f(x) \sim \dfrac{1}{x}$ as $x \to +0$.*
- *As $x \to +\infty$, the bounds*

$$\frac{1}{2}(f(x) + f(x+1)) \leq f(x) \leq \frac{1}{2}(f(x) + f(x-1))$$

with $\frac{1}{x} \sim \frac{1}{x-1}$ gives $f(x) \sim \dfrac{1}{2x}$.

f) *Let $x > 0$. For all $t \in [0,1]$ and all integers N, $\dfrac{1}{1+t} = \sum_{n=0}^{N} (-t)^n + \dfrac{(-t)^{N+1}}{1+t}$, hence for all $t \in [0,1]$,*

$$\frac{t^{x-1}}{1+t} = \sum_{n=0}^{N} (-1)^n t^{n+x-1} + (1)^{N+1} \frac{t^{N+x}}{1+t}.$$

The function $t \mapsto \dfrac{t^{x-1}}{1+t}$ *being integrable on* $[0,1]$ *(since* $x > 0$*), we obtain by integrating:*

$$\int_0^1 \frac{t^{x-1}}{1+t}\,\mathrm{d}t = \sum_{n=0}^{N} \frac{(-1)^n}{n+x} + (-1)^{N+1} \int_0^1 \frac{t^{N+x}}{1+t}\,\mathrm{d}t,$$

and since $\left|(-1)^{N+1} \int_0^1 \dfrac{t^{N+x}}{1+t}\,\mathrm{d}t\right| \leq \int_0^1 t^{N+x}\,\mathrm{d}t = \dfrac{1}{N+x+1} \to 0$ *as* $N \to +\infty$*, we obtain the desired result by taking the limit as* $N \to +\infty$*.*

Chapter 3

Power Series

Power series are a particular class of function series. We will first focus on the properties of the sum of a power series (such as the domain of convergence, continuity,...). Then, we will explore how to express common functions as sums of power series.

Definition 3.0.1 *Let (a_n) be a sequence of complex numbers.*

A power series in the complex variable z with coefficients (a_n) is defined as the series of functions $\sum\limits_{n=0}^{+\infty} a_n z^n$.

The domain of convergence of this series is given by

$$D = \{z \in \mathbb{C} \mid \sum_{n=0}^{+\infty} a_n z^n \text{ converges}\};$$

this is the domain of definition of the sum of the power series, defined by:

$$\forall z \in D, \quad f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

Remark 3.0.2 *Any series of the form $\sum\limits_{n=0}^{\infty} a_n z^n$ converges at $z = 0$. Indeed, let $(S_n)_n$ be the sequence of partial sums of the series $\sum\limits_{n=0}^{\infty} a_n z^n$. Then, we have*

$$S_n(z) = \sum_{k=0}^{n} a_k z^k \Longrightarrow S_n(0) = a_0 \Longrightarrow \lim_{n\to+\infty} S_n(0) = a_0.$$

Thus, the domain of convergence of the series $\sum\limits_{n=0}^{\infty} a_n z^n$ is never empty, because we always have $0 \in D$ and $f(0) = a_0$.

The two basic examples of power series are the geometric series and the exponential series. We will later see that many common series can be expressed in terms of one or the other.

Geometric series:

$$\forall z, \ |z| < 1, \quad \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n = 1 + z + z^2 + ... + z^n + ...$$

Exponential series:

$$\forall z \in \mathbb{C}, \quad \exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + ... + \frac{z^n}{n!} + \dots$$

The calculation of the sum of the geometric series is straightforward, thanks to the explicit formula for the partial sums. However, the fact that the sum of the exponential series is $\exp(z)$ is not immediately obvious. Two approaches can be taken to understand this.

1. After proving that the series converges for all z, we can define the complex exponential as the sum of this series. Based on the results we will establish regarding power series, we can then prove all the classical properties of the exponential function.

2. There are other definitions of the exponential function. For instance, it can be defined on $\mathbb{R}$ as the inverse function of the natural logarithm, which is itself defined as the antiderivative of $\frac{1}{x}$ that vanishes at 1; this definition can then be extended to the entire $\mathbb{C}$. We can subsequently prove that $\exp(z)$ is the sum of the exponential series.

Another class of power series is given by the example below:

Examples 3.0.3

1. *A polynomial in z is a power series whose coefficients vanish after a certain degree. In this case, $D = \mathbb{C}$.*

2. $\sum_{n\in\mathbb{N}} n!z^n$ *is a power series for which $D = \{0\}$ I.Indeed, To find the domain of convergence for the series $\sum_{n=0}^{\infty} n!z^n$, we can use the* ***ratio test****. The ratio test states that a series $\sum_{n=0}^{\infty} a_n$ converges absolutely if*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$$

Step 1: Identify a_n

For our series, we have:

$$a_n = n!z^n.$$

Step 2: Compute $\frac{a_{n+1}}{a_n}$

We calculate:

$$a_{n+1} = (n+1)!z^{n+1} = (n+1)n!z^{n+1}.$$

Thus,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!z^{n+1}}{n!z^n} = (n+1)z.$$

Step 3: Apply the Ratio Test

Now, we take the limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |(n+1)z| = \lim_{n\to\infty} (n+1)|z| = \infty \quad \textit{for any } z \neq 0.$$

This limit diverges for any $z \neq 0$.

Step 4: Conclusion

- ***If*** $z = 0$*: The series becomes* $\sum_{n=0}^{\infty} n! \cdot 0^n = 0$*, which converges.*
- ***If*** $z \neq 0$*: The series diverges.*

Thus, the domain of convergence for the series $\sum_{n=0}^{\infty} n! z^n$ *is:*

$$\textit{The series converges only at } z = 0.$$

3. *If a is a non-zero complex number,* $\sum_{n\in\mathbb{N}} \frac{z^n}{a^n}$ *is a power series for which* $D = \{z \in \mathbb{C} \text{ tq } |z| < |a|\}$ *is the open disk centered at 0 with radius $|a|$ (and for all $z \in D$, we have* $f(z) = \frac{a}{a-z}$*).*

4. *If a is a non-zero complex number,* $\sum_{n\in\mathbb{N}} \frac{z^n}{n^2 a^n}$ *is a power series for which* $D = \{z \in \mathbb{C} \text{ tq } |z| \leq |a|\}$ *is the closed disk centered at 0 with radius $|a|$.*

3.1 Radius of Convergence of a Power Series

3.1.1 Definition of the Radius of Convergence and Properties

This section discusses the properties of the radius of convergence for power series. Understanding these properties is crucial for analyzing the behavior of power series within their convergence regions. All results presented here are founded on the following important theorem:

Theorem 3.1.1 (Abel's Lemma) *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series.*

We assume that there exists a non-zero complex number z_0 such that the sequence $(a_n z_0^n)$ is bounded.

Then, for any complex number z such that $|z| < |z_0|$, the infinite series $\sum_{n\in\mathbb{N}} a_n z^n$ *is absolutely convergent.*

Proof: By hypothesis: $\exists M \in \mathbb{R}_+ \text{ tq } \forall n \in \mathbb{N}\ ,\ |a_n z_0^n| \leq M$.

Since $z_0 \neq 0$, we will have then, for any $z \in \mathbb{C}$: $|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \leq M \left|\frac{z}{z_0}\right|^n$. Now, if $|z| < |z_0|$, the geometric series $\sum_{n\in\mathbb{N}} \left|\frac{z}{z_0}\right|^n$ is convergent, hence the result according to the comparison theorems for series with positive terms. □

Remark 3.1.2

1. *If there exists a $z_0 \in \mathbb{C}^*$ such that the series* $\sum_{n\in\mathbb{N}} a_n z_0^n$ *is convergent, then the sequence $(a_n z_0^n)$ tends to 0 when $n \to +\infty$, hence this sequence is bounded. Then, according to Abel's lemma, for any z such that $|z| < |z_0|$, the power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is absolutely convergent, hence convergent.*
2. *If there exists a $z_0 \in \mathbb{C}^*$ such that the sequence $(a_n z_0^n)$ is not bounded, then, for any z such that $|z| \geq |z_0|$ the sequence $(a_n z^n)$ is not bounded, hence the power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *diverges.*

Therefore, the key to studying a power series is to find the values of z for which the sequence $(a_n z^n)$ is bounded.

Definition 3.1.3 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series. The set* I *of positive real numbers* r *such that the sequence* $(a_n r^n)$ *is bounded is an interval containing* 0 *(indeed,* $0 \in I$*, and if* $r > 0$ *belongs to* I*, then any real number* r' *with* $0 \leq r' \leq r$ *also belongs to* I*).*
We then call the radius of convergence R *of this power series the least upper bound (in* $\overline{\mathbb{R}}$*) of this interval:*

$$R = \sup\{r \in \mathbb{R}_+ \mid (a_n r^n) \ \text{bounded}\} \ \text{or} \ R = \sup\{r \in \mathbb{R}_+ \mid (|a_n|\, r^n) \ \text{bounded}\} \quad (R \in \mathbb{R}_+ \cup \{+\infty\})\,.$$

Recall that any bounded subset of $\mathbb{R}$ has a finite least upper bound. By convention, the least upper bound of an unbounded subset is $+\infty$.

Remark 3.1.4 *There are two important special cases.*

1. *If* $R = 0$*, for any non-zero* z*, the sequence* $(a_n z^n)$ *is not bounded: the series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is then grossly divergent. It only converges for* $z = 0$*.*

   Example : $\sum_{n\in\mathbb{N}} n! z^n$.

2. *If* $R = +\infty$*, the sequence* $(a_n r^n)$ *is bounded for any* $r \in \mathbb{R}_+$*. Since, for any complex number* z*, there exists* $r \in \mathbb{R}_+$ *such that* $|z| < r$*, it follows from Abel's lemma that the power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is absolutely convergent for any* $z \in \mathbb{C}$*.*

   Example : $\sum_{n\in\mathbb{N}} \frac{z^n}{n!}$.

In the general case, we have the following result:

Theorem 3.1.5 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series with radius of convergence* R*.*

1. *If* $R > 0$*, then:*
   for any z *such that* $|z| < R$*, the series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is absolutely convergent.*

2. *If* $R < +\infty$ *then:*
   for any z *such that* $|z| > R$*, the series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is (grossly) divergent.*

Proof:

1. If $|z| < R$ then, by definition of the least upper bound, there exists r such that $|z| < r \leq R$ and such that the sequence $(a_n r^n)$ is bounded. The result then follows directly from Abel's lemma.

2. If $|z| > R$ then, by definition of the least upper bound, the sequence $(a_n z^n)$ is not bounded. The series $\sum_{n\in\mathbb{N}} a_n z^n$ is therefore grossly divergent.

□

Other characterizations of the radius of convergence may be useful:

Proposition 3.1.6 *Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a power series. Its radius of convergence R can be defined by one of the following equalities:*

a) $R = \sup\left\{|z| \text{ tq the series } \sum_{n\in\mathbb{N}} a_n z^n \text{ is absolutely convergent}\right\}$;

b) $R = \sup\left\{|z| \text{ tq the series } \sum_{n\in\mathbb{N}} a_n z^n \text{ is convergent}\right\}$;

c) $R = \sup\left\{|z| \text{ tq } \lim_{n\to+\infty} a_n z^n = 0\right\}$;

d) $R = \sup\{|z| \text{ tq the sequence } (a_n z^n) \text{ is bounded}\}$.

Proof: Let us denote $E_a = \left\{|z| \text{ tq the series } \sum_{n\in\mathbb{N}} a_n z^n \text{ is absolutely convergent}\right\}$ and $R_a = \sup E_a$; similarly, let E_b, E_c and E_d be the other three sets in the statement, and R_b, R_c and R_d their least upper bounds (in $\overline{\mathbb{R}}$).

- By definition, the radius of convergence of the power series is $R = R_d$.
- Since $E_a \subset E_b \subset E_c \subset E_d$, we have $R_a \leq R_b \leq R_c \leq R_d$.
- If $R_d = 0$ or $R_d = +\infty$, we are done (cf. the previous remarks). Otherwise, we still need to show $R_d \leq R_a$.

  By definition of the least upper bound of E_d, for any $\varepsilon > 0$ there exists $z_0 \in E_d$ such that $R_d - \varepsilon < |z_0| \leq R_d$. Since $R_d > 0$ we can assume ε is small enough so that $R_d - \varepsilon > 0$; let then $z \in \mathbb{C}$ such that $|z| = R_d - \varepsilon$. Since $z_0 \in E_d$ the sequence $(a_n z_0^n)$ is bounded, so according to Abel's lemma, since $|z| < |z_0|$, the series $\sum_{n\in\mathbb{N}} a_n z^n$ is absolutely convergent, that is, $|z| \in E_a$.
  It follows that $|z| \leq R_a$ that is $R_d - \varepsilon \leq R_a$; this being true for any $\varepsilon > 0$ small enough, we deduce that $R_d \leq R_a$.

□

Definition 3.1.7 *Let $\sum_{n\in\mathbb{N}} a_n z^n$ be a power series with radius of convergence $R > 0$.*
We call the <u>open disc of convergence</u> of this power series the set

$$D = \{z \in \mathbb{C} \text{ tq } |z| < R\} \text{ (the open ball centered at 0 with radius } R).$$

Remark 3.1.8 *In the case of a power series $\sum_{n\in\mathbb{N}} a_n x^n$ of the real variable x, we then speak of the <u>open interval of convergence</u> $]-R, R[$.*

This open disc of convergence is therefore characterized by the following properties:

- If $z \in D$, the power series $\sum_{n\in\mathbb{N}} a_n z^n$ is absolutely convergent.
- If $z \notin \overline{D}$, the power series $\sum_{n\in\mathbb{N}} a_n z^n$ is (grossly) divergent.
- If $|z| = R$ we cannot say anything *a priori* (the circle centered at 0 with radius R is sometimes called the *circle of uncertainty*).

Remark 3.1.9 *On the circle of uncertainty (let us denote it $\mathcal{C}$), three situations are actually possible:*

- *there exists $z \in \mathcal{C}$ such that the series $\sum\limits_{n\in\mathbb{N}} a_n z^n$ diverges grossly; then this series diverges grossly on the entire circle.*

  Examples : $\sum\limits_{n\in\mathbb{N}} nz^n$, $\sum\limits_{n\in\mathbb{N}} z^n$.

- *there exists $z \in \mathcal{C}$ such that the series $\sum\limits_{n\in\mathbb{N}} a_n z^n$ is convergent; we cannot then say anything a priori for the other points of the circle.*

  Example : $\sum\limits_{n\in\mathbb{N}^*} \dfrac{z^n}{n}$.

- *there exists $z \in \mathcal{C}$ such that the series $\sum\limits_{n\in\mathbb{N}} a_n z^n$ is absolutely convergent; then this series converges absolutely on the entire circle.*

  Example : $\sum\limits_{n\in\mathbb{N}^*} \dfrac{z^n}{n^2}$.

Proposition 3.1.10 *For any real number α, the series*

$$\sum n^\alpha z^n$$

has radius of convergence $R = 1$.

Proof: Indeed, $n^\alpha r^n$ tends to 0 for $r < 1$, to $+\infty$ for $r > 1$. The series $\sum n^\alpha z^n$ converges for $|z| < 1$, diverges for $|z| > 1$. Now consider a complex number z of modulus 1 : $z = e^{i\theta}$

- If $\alpha \geq 0$, the series $\sum n^\alpha e^{in\theta}$ diverges.

- If $\alpha < -1$, the series $\sum n^\alpha e^{in\theta}$ is absolutely convergent.

- If $-1 \leq \alpha < 0$, the series $\sum n^\alpha e^{in\theta}$ is convergent for $\theta \neq 2k\pi$, but not absolutely convergent. For $z = 1$, the series $\sum n^\alpha$ diverges.

□

3.1.2 Methods to Calculate the Radius of Convergence

Application of the Definition of R

The following remarks follow directly from the definition of the radius of convergence of a power series (or one of the equivalent definitions given above) and can assist in its determination (or in establishing bounds for it).

Let $\sum\limits_{n\in\mathbb{N}} a_n z^n$ be a power series and R its radius of convergence.

- If the series $\sum a_n z^n$ converges for $z = z_0$, then $R \geqslant |z_0|$.

- If the series $\sum a_n z^n$ diverges for $z = z_1$ then $R \leqslant |z_1|$.

- If the series $\sum a_n z^n$ converges non-absolutely for $z = z_2$, then $R = |z_2|$.

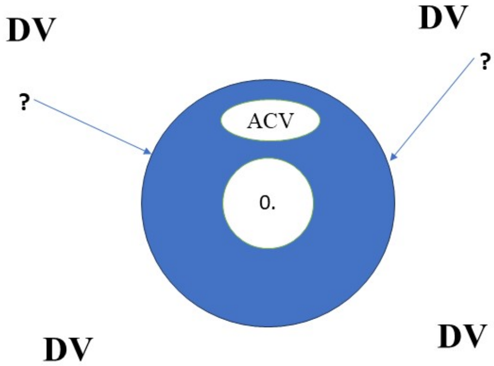


Figure 3.1: Disc of convergence of a power series.

- If the sequence $(a_n z^n)$ is bounded for $z = z_0$, then $R \geqslant |z_0|$.
- If the sequence $(a_n z^n)$ does not converge to 0 for $z = z_1$, then $R \leqslant |z_1|$.
- If the sequence $(a_n z^n)$ is bounded but does not converge to 0 for $z = z_2$, then $R = |z_2|$.

Examples 3.1.11

1. $\sum_{n\in\mathbb{N}^*} \frac{z^n}{n}$ *is conditionally convergent for* $z = -1$ *: its radius of convergence is therefore* $R = 1$.
We can also say: the series diverges for $z = 1$*, so* $R \leq 1$*; and it converges absolutely for* $|z| < 1$ *(by comparison with the geometric series with general term* $|z|^n$*), so* $R \geq 1$.

2. $\sum_{n\in\mathbb{N}^*} \frac{z^n}{n^2}$ *converges (absolutely) for* $z = 1$ *: its radius of convergence* R *is therefore greater than or equal to 1; since the sequence* $\left(\frac{z^n}{n^2}\right)$ *does not converge to* 0 *as long as* $|z| > 1$*, we have:* $R \leqslant 1$*. Ultimately,* $R = 1$.

3. $\sum_{n\in\mathbb{N}} \frac{z^n}{(3+(-1)^n)^n}$ *:*

$\left|\frac{z^n}{(3+(-1)^n)^n}\right| \leqslant \frac{|z|^n}{2^n}$ *: this sequence is therefore bounded for* $z = 2$*, but it does not converge to* 0*, so* $R = 2$.

The first two examples are a special case of Proposition 3.1.10, which is now part of the curriculum, so can be used directly.

The radius of convergence of the series $\sum_{n\in\mathbb{N}} a_n z^n$ is related to the coefficients a_n in the following way.

Using Comparison Rules

Theorem 3.1.12 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *and* $\sum_{n\in\mathbb{N}} b_n z^n$ *be two power series with respective radii of convergence* R_a *and* R_b.

1. *If* $a_n \underset{n\to+\infty}{\sim} b_n$, *then* $R_a = R_b$.

2. *If* $|a_n| \leq |b_n|$ *(at least from a certain rank), then* $R_a \geq R_b$.

3. *If* $a_n = O(b_n)$ *or* $a_n = o(b_n)$, *then* $R_a \geq R_b$.

Proof:

1. If $a_n \underset{n\to+\infty}{\sim} b_n$, then $(a_n z^n)$ is bounded $\iff$ $(b_n z^n)$ is bounded, hence the equality of the radii of convergence.

2. Suppose $|a_n| \leq |b_n|$. For any z such that $|z| < R_b$, the series $\sum_{n\in\mathbb{N}} |b_n|\,|z|^n$ converges, so since $0 \leq |a_n|\,|z|^n \leq |b_n|\,|z|^n$, the series $\sum_{n\in\mathbb{N}} |a_n|\,|z|^n$ also converges, so $|z| \leq R_a$.

   Thus: $\forall\, z \in \mathbb{C}$, $|z| < R_b \Longrightarrow |z| \leq R_a$. This implies $R_b \leq R_a$.

3. Immediate consequence of the previous result since $a_n = O(b_n)$ can be written as: $\exists M \in \mathbb{R}_+$ tq $|a_n| \leq M\,|b_n|$.

□

Examples 3.1.13

1. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *where* a_n *denotes the n-th decimal after the comma in the infinite decimal expansion of* π.
We have $a_n \leq 9$ *for any* n, *so the radius of convergence* R *of the power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *is greater than that of the power series* $\sum_{n\in\mathbb{N}} 9z^n$, *which is equal to* 1 *(geometric series with common ratio* z*):*

$$R \geq 1.$$

Furthermore, the infinite series $\sum_{n\in\mathbb{N}} a_n$ *diverges since its general term does not tend to* 0*: indeed, if the sequence* (a_n) *tended to* 0*, since it is a sequence of integers, it would be constant equal to* 0 *from a certain rank, and the number* π *would be a decimal number....*
Since the power series $\sum_{n\in\mathbb{N}} a_n z^n$ *diverges for* $z = 1$ *we have*

$$R \leq 1$$

and finally, $R = 1$.

2. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}^*} d_n z^n$ *where* d_n *is the number of divisors of* n.
Easily, we have $1 \leq d_n \leq n$ *for any* $n \in \mathbb{N}^*$, *so the radius of convergence* R *of the power series* $\sum_{n\in\mathbb{N}^*} d_n z^n$ *lies between that of the power series* $\sum z^n$, *which is* 1, *and that of the power series* $\sum nz^n$, *which is also* 1 *(cf. Proposition 3.1.10).*
Therefore, $R = 1$.

3. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}}(\cos n)z^n$*. What is its sum?*

- *Since* $|\cos n| \leq 1$*, the radius of convergence* R *of the power series* $\sum_{n\in\mathbb{N}}(\cos n)z^n$ *is greater than that of the power series* $\sum z^n$*, that is,* $R \geq 1$*.*

  We know (classical exercise) that the sequence $(\cos n)$ *is divergent; it follows that the series* $\sum \cos n$ *diverges, so* $R \leq 1$*.*

  In conclusion, $R = 1$*.*

- *For* $|z| < 1$ *we have :* $\sum_{n=0}^{+\infty} \mathrm{e}^{\mathrm{i}n} z^n = \frac{1}{1-\mathrm{e}^{\mathrm{i}}z}$ *(geometric series with common ratio* $z\mathrm{e}^{\mathrm{i}}$ *with* $\left|z\mathrm{e}^{\mathrm{i}}\right| < 1$*), and also* $\sum_{n=0}^{+\infty} \mathrm{e}^{-\mathrm{i}n} z^n = \frac{1}{1-\mathrm{e}^{-\mathrm{i}}z}$*.*

  We deduce:

$$\sum_{n=0}^{+\infty}(\cos n)z^n = \frac{1}{2}\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}}z} + \frac{1}{1-\mathrm{e}^{-\mathrm{i}}z}\right) = \ldots \qquad \textit{(to be arranged)}$$

4. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}}(\operatorname{ch} n)z^n$*. What is its sum?*

Since $\operatorname{ch} n \underset{n\to+\infty}{\sim} \frac{1}{2}\mathrm{e}^n$*, the radius of convergence* R *of the power series* $\sum_{n\in\mathbb{N}}(\operatorname{ch} n)z^n$ *is equal to that of the power series* $\sum_{n\in\mathbb{N}} \mathrm{e}^n z^n$*, that is,* $R = \frac{1}{\mathrm{e}}$ *(the geometric series with common ratio* $\mathrm{e}z$ *converges if and only if* $|\mathrm{e}z| < 1$*).*

For $|z| < \frac{1}{\mathrm{e}}$*, we can write*

$$\sum_{n=0}^{+\infty}(\operatorname{ch} n)z^n = \frac{1}{2}\left(\sum_{n=0}^{+\infty}\mathrm{e}^n z^n + \sum_{n=0}^{+\infty}\mathrm{e}^{-n}z^n\right)$$

since both series converge. *The desired sum is therefore equal to* $\frac{1}{2}\left(\frac{1}{1-\mathrm{e}z} + \frac{1}{1-\frac{z}{\mathrm{e}}}\right)$.

The Cauchy-Hadamard Rule

The following result is Cauchy-Hadamard's theorem, which is used to determine the radius of convergence of a power series.

Theorem 3.1.14 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series and let* R *be its radius of convergence. Assume that there exists* $l = \lim_{n\to+\infty} \sqrt[n]{|a_n|}$*. Then,*

1. *if* $0 < l < +\infty$ *then* $R = \frac{1}{l}$
2. *if* $l = 0$ *then* $R = +\infty$
3. *if* $l = \infty$ *then* $R = 0$*.*

Proof:

1. Assume that $0 < l < +\infty$, where $l = \lim_{n\to+\infty} \sqrt[n]{|a_n|}$. And let $L = \lim_{n\to+\infty} \sqrt[n]{|a_n z^n|}$, hence

$$L = \lim_{n\to+\infty} |z| \sqrt[n]{|a_n|} = |z| \lim_{n\to+\infty} \sqrt[n]{|a_n|} = |z|l.$$

   Thus,

$$L = |z|l.$$

   The Cauchy rule for infinite series then allows us to conclude: if $L < 1 \iff |z| < \frac{1}{l}$ then $\sum_{n\in\mathbb{N}} a_n z^n$ converges absolutely on $D_{\frac{1}{l}}$ and hence $R = \frac{1}{l}$. And it diverges if $|z| > \frac{1}{l}$.

2. If $l = 0$, then $L = 0$, according to the Cauchy rule the series $\sum_{n\in\mathbb{N}} a_n z^n$ converges for all $z \in \mathbb{C}$. Therefore $R = +\infty$.

3. If $l = +\infty$, then $L = +\infty$. So by the Cauchy rule, the series $\sum_{n\in\mathbb{N}} a_n z^n$ diverges for all $z \neq 0$. Therefore $R = 0$.

□

Example 3.1.15 *Calculate the radius of convergence of the series* $\sum_{n\geq 2}(3^n \ln n)z^n$.
We have $a_n = 3^n \ln n$, *and therefore the radius of convergence* R *is:*

$$R = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{|a_n|}} = \frac{1}{3\lim_{n\to+\infty}(\ln n)^{\frac{1}{n}}} = \frac{1}{3}.$$

Using D'Alembert's Rule for Infinite Series

Here we consider a power series $\sum_{n\in\mathbb{N}} a_n z^n$ such that, for any integer n, $a_n \neq 0$.
For any $z \in \mathbb{C}^*$, the sequence with general term $u_n = a_n z^n$ therefore does not vanish. We can then try to apply D'Alembert's rule to the series with positive terms $\sum |u_n|$, by studying the eventual limit when $n \to +\infty$ of the ratio $\frac{|u_{n+1}|}{|u_n|}$, i.e. the expression $\left|\frac{a_{n+1}}{a_n}\right| |z|$.

Recall that, if $\ell = \lim_{n\to+\infty} \frac{|u_{n+1}|}{|u_n|}$ exists (in $\overline{\mathbb{R}}$):

- if $\ell < 1$, the series with general term u_n is absolutely convergent;
- and if $\ell > 1$, this series is (grossly) divergent.

The following theorem is due to Hadamard-d'Alembert. It also allows us to determine the radius of convergence.

Theorem 3.1.16 (Hadamard-d'Alembert Theorem) *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series and let* R *be its radius of convergence. Assume that there exists* $l = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$. *Then,*

1. *if* $0 < l < +\infty$ *then* $R = \frac{1}{l}$
2. *if* $l = 0$ *then* $R = +\infty$
3. *if* $l = \infty$ *then* $R = 0$.

Proof: The proof is identical to that given in Cauchy-Hadamard's theorem (see Theorem 3.1.14), except that here we use D'Alembert's criterion instead of Cauchy's. □

Examples 3.1.17

1. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}^*} \frac{n!}{n^n} z^n$.

   For $n \in \mathbb{N}^*$ *and* $z \neq 0$, *let* $u_n = \frac{n!}{n^n} z^n$. *Then*

   $$\frac{|u_{n+1}|}{|u_n|} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} |z| = \left(\frac{n}{n+1}\right)^n |z| .$$

   Now (classical calculation, already done and to be known):

   $$\lim_{n\to+\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = \lim_{n\to+\infty} \mathrm{e}^{n\ln\left(1+\frac{1}{n}\right)} = \mathrm{e}$$

   so $\lim_{n\to+\infty} \frac{|u_{n+1}|}{|u_n|} = \frac{|z|}{\mathrm{e}}$. *D'Alembert's rule for infinite series then allows us to say that:*

   - *if* $|z| < \mathrm{e}$, *the series with general term* u_n *is (absolutely) convergent;*
   - *and if* $|z| > \mathrm{e}$, *it diverges.*

   In conclusion, the desired radius of convergence is equal to e.

2. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}} \frac{n}{2^n} z^{3n}$.

   Let $u_n = \frac{n}{2^n} z^{3n}$. *For* $z \neq 0$ *we have*

   $$\frac{|u_{n+1}|}{|u_n|} = \frac{n+1}{2^{n+1}} \frac{2^n}{n} \frac{\left|z^{3(n+1)}\right|}{|z^{3n}|} = \frac{n+1}{2n} |z|^3 .$$

   Therefore, $\lim_{n\to+\infty} \frac{|u_{n+1}|}{|u_n|} = \frac{|z|^3}{2}$. *According to D'Alembert's rule for infinite series:*

   - *if* $|z|^3 < 2$, *that is, if* $|z| < \sqrt[3]{2}$, *the series with general term* u_n *is (absolutely) convergent;*
   - *and if* $|z| > \sqrt[3]{2}$, *it diverges.*

   In conclusion, the desired radius of convergence is equal to $\sqrt[3]{2}$.

3. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}} n! z^{n^2}$.

   Let $u_n = n! z^{n^2}$. *For* $z \neq 0$ *we have*

   $$\frac{|u_{n+1}|}{|u_n|} = (n+1) |z|^{(n+1)^2-n^2} = (n+1) |z|^{2n+1} .$$

   Therefore, $\lim_{n\to+\infty} \frac{|u_{n+1}|}{|u_n|} = \begin{cases} 0 & \textit{if } |z| < 1 \\ +\infty & \textit{otherwise} \end{cases}.$

   According to D'Alembert's rule for infinite series, if $|z| < 1$ *the series with general term* u_n *is (absolutely) convergent, and otherwise it is divergent.*

   In conclusion, the desired radius of convergence is equal to 1.

4. *Radius of convergence of the power series* $\sum_{n\in\mathbb{N}} \binom{2n}{n} z^{2n+1}$.

Let $u_n = \binom{2n}{n} z^{2n+1}$. *For* $z \neq 0$ *we have*

$$\frac{|u_{n+1}|}{|u_n|} = 2\frac{2n+1}{n+1}|z|^2$$

so $\lim_{n\to+\infty} \frac{|u_{n+1}|}{|u_n|} = 4|z|^2$. *According to D'Alembert's rule for infinite series:*

- *if* $|z| < \frac{1}{2}$, *the series with general term* u_n *is (absolutely) convergent;*
- *and if* $|z| > \frac{1}{2}$, *it diverges.*

In conclusion, the radius of convergence of this series is equal to $\frac{1}{2}$.

Remark : another possible solution is to use Stirling's formula...

5. *Calculate the radius of convergence of the series* $\sum_{n\geq 0} \frac{z^n}{2n+5}$.

We have $a_n = \frac{z^n}{2n+5}$, *so*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2n+5}{2n+7}\right| = 1.$$

Thus, the radius of convergence of this series is

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = 1.$$

Remark 3.1.18 *The radius of convergence of a power series* $\sum_{n\in\mathbb{N}} a_n z^n$ *depends only on the modulus of the coefficients* a_n *(i.e.,* $|a_n|$*). Then, the series* $\sum_{n\in\mathbb{N}} a_n z^n$, $\sum_{n\in\mathbb{N}} \overline{a_n} z^n$ *and* $\sum_{n\in\mathbb{N}} |a_n| z^n$ *have the same radius of convergence.*

Remember that a power series converges absolutely on its disc of convergence. Moreover, the convergence is uniform on any closed disc included in the disc of convergence.

Proposition 3.1.19 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series, with radius of convergence* R. *Let* r *be a real number such that* $0 < r < R$.

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N},\ n \geq n_0,\ \forall z \ s.t.\ |z| \leq r,\quad \left|\sum_{k=n+1}^{+\infty} a_n z^n\right| < \varepsilon.$$

Proof: Let us fix r' such that $r < r' < R$. For any $n \in \mathbb{N}$:

$$|a_n z^n| \leq |a_n|(r')^n \frac{r^n}{(r')^n} \leq M\frac{r^n}{(r')^n},$$

where M is an upper bound of $|a_n|(r')^n$ (which exists by definition of the radius of convergence). Then, for any complex number z with modulus less than or equal to r:

$$\left|\sum_{k=n+1}^{+\infty} a_n z^n\right| \leq \sum_{k=n+1}^{+\infty} M\frac{r^k}{(r')^k} = \frac{Mr}{r'-r}\left(\frac{r}{r'}\right)^{n+1}$$

This bound being independent of z, the convergence is indeed uniform. □

3.2 Operations on Power Series

The results of this section have numerous practical implications for calculating sums of power series. We analyze the behavior of series with respect to standard operations (linear combinations and products).

3.2.1 Sum of Two Power Series

Theorem 3.2.1 *Let $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$ be two power series with respective radii of convergence R_a and R_b.*

- *If $R_a \neq R_b$, the power series $\sum_{n\in\mathbb{N}}(a_n+b_n)z^n$ has radius of convergence $R = \min(R_a, R_b)$.*
- *If $R_a = R_b$, the power series $\sum_{n\in\mathbb{N}}(a_n+b_n)z^n$ has radius of convergence $R \geq R_a$.*
- *In both cases, for any z such that $|z| < \min(R_a, R_b)$:*

$$\sum_{n=0}^{+\infty}(a_n+b_n)z^n = \sum_{n=0}^{+\infty} a_n z^n + \sum_{n=0}^{+\infty} b_n z^n .$$

Proof: If z is a complex number such that $|z| < \min(R_a, R_b)$, the two series $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$ converge, so $\sum_{n\in\mathbb{N}}(a_n+b_n)z^n$ converges; we deduce that $R \geqslant \min(R_a, R_b)$.
Suppose for example $R_a < R_b$; if z is a complex number such that $R_a < |z| < R_b$, $\sum_{n\in\mathbb{N}} a_n z^n$ diverges and $\sum_{n\in\mathbb{N}} b_n z^n$ converges, so $\sum_{n\in\mathbb{N}}(a_n+b_n)z^n$ diverges; thus $R = R_a = \min(R_a, R_b)$. $\square$

Remark 3.2.2 *If $R_a = R_b$, it is possible that $R > \min(R_a, R_b)$: for example, $\sum_{n\in\mathbb{N}} z^n$ and $\sum_{n\in\mathbb{N}} (\frac{1}{2^n} - 1)z^n$ both have a radius of convergence equal to 1, but the radius of convergence of their sum is 2.*

3.2.2 Cauchy Product of Two Power Series

Here we consider two power series $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$ with respective radii of convergence R_a and R_b.
For z such that $|z| < \min(R_a, R_b)$, let $u_n = a_n z^n$ and $v_n = b_n z^n$. The two series with general term u_n and v_n are then absolutely convergent. We can then consider their *Cauchy product series* whose general term w_n is defined by:

$$w_n = \sum_{k=0}^{n} u_k v_{n-k} \quad \text{so} \quad w_n = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n .$$

If we let, for any natural integer n:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{\substack{p,q\in\mathbb{N} \\ p+q=n}} a_p b_q ,$$

the power series $\sum_{n\in\mathbb{N}} c_n z^n$ is called the Cauchy product series of the two power series $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$. The results from the course on absolutely convergent series then allow us to directly state the following theorem.

Theorem 3.2.3 *Let $\sum_{n\in\mathbb{N}} a_n z^n$ and $\sum_{n\in\mathbb{N}} b_n z^n$ be two power series with respective radii of convergence R_a and R_b.*
Let, for any $n \in \mathbb{N}$:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

The radius of convergence R_c of the power series $\sum_{n\in\mathbb{N}} c_n z^n$ is such that $R_c \geq \min(R_a, R_b)$. Moreover, for any complex number z such that $|z| < \min(R_a, R_b)$, we have:

$$\sum_{n=0}^{+\infty} c_n z^n = \left(\sum_{n=0}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right).$$

Examples 3.2.4

1. *By performing the Cauchy product of the power series $\sum_{n\in\mathbb{N}} z^n$ by itself, we obtain:*

$$\text{for any complex number } z \text{ such that } |z| < 1 \ , \ \frac{1}{(1-z)^2} = \sum_{n=0}^{+\infty}(n+1)z^n \, .$$

Indeed, we have for any z such that $|z| < 1$, $\dfrac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$. It is therefore sufficient to apply the previous formula with $a_n = b_n = 1$.

2. *More generally, we can prove by induction on the integer $p \in \mathbb{N}^*$ that:*

$$\forall p \in \mathbb{N}^* \ , \ \forall z \in \mathbb{C} \ tq \ |z| < 1 \ , \ \frac{1}{(1-z)^p} = \sum_{n=0}^{+\infty} \binom{n+p-1}{p-1} z^n \, .$$

   (a) *The proposed formula is verified for $p = 1$ (power series expansion of $\dfrac{1}{1-z}$) and also for $p = 2$ according to the previous example.*

   (b) *Suppose it is verified at rank p, and let us check it at rank $p+1$.*

   (c) 1st solution: using the Cauchy product
   Given the induction hypothesis, we have, for any z such that $|z| < 1$:

$$\frac{1}{(1-z)^p} = \sum_{n=0}^{+\infty} \binom{n+p-1}{p-1} z^n \qquad \text{and also} \qquad \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n.$$

   By defining $a_n = \dbinom{n+p-1}{p-1}$ and $b_n = 1$, the previous theorem directly gives, for any z such that $|z| < 1$:

$$\frac{1}{(1-z)^{p+1}} = \sum_{n=0}^{+\infty} c_n z^n \qquad \text{where } \ c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

We then calculate: $c_n = \sum_{k=0}^{n} \binom{k+p-1}{p-1}$*. However, according to Pascal's triangle formula, for any* $k \in \mathbb{N}$ *(with the standard conventions for binomial coefficients), we have:*

$$\binom{k+p-1}{p-1} + \binom{k+p-1}{p} = \binom{k+p}{p}$$

so, by telescoping:

$$\begin{aligned} c_n &= \sum_{k=0}^{n} \left[\binom{k+p}{p} - \binom{k+p-1}{p} \right] \\ &= \binom{n+p}{p} - \underbrace{\binom{p-1}{p}}_{=0} \\ &= \binom{n+p}{p}. \end{aligned}$$

(we have just proven above the generalized Pascal's triangle formula).
We have therefore proven, for any z *such that* $|z| < 1$ *:*

$$\frac{1}{(1-z)^{p+1}} = \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n$$

which is the desired formula at order $p+1$*.*

(d) 2nd solution, faster

We want to prove, for any z *such that* $|z| < 1$*, the equality:* $\frac{1}{(1-z)^{p+1}} = \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n$*, which is equivalent to proving that:*

$$(1-z) \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n = \frac{1}{(1-z)^p}.$$

And this follows from the calculation below:

$$\begin{aligned} (1-z) \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n &= \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n - \sum_{n=0}^{+\infty} \binom{n+p}{p} z^{n+1} \\ &= \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n - \sum_{n=1}^{+\infty} \binom{n+p-1}{p} z^n \quad \textit{(change of index)} \\ &= \sum_{n=0}^{+\infty} \binom{n+p}{p} z^n - \sum_{n=0}^{+\infty} \binom{n+p-1}{p} z^n \qquad \textit{(since } \binom{p-1}{p} = 0\textit{)} \\ &= \sum_{n=0}^{+\infty} \left[\binom{n+p}{p} - \binom{n+p-1}{p} \right] z^n \\ &= \sum_{n=0}^{+\infty} \binom{n+p-1}{p-1} z^n \qquad \textit{(Pascal's triangle)} \\ &= \frac{1}{(1-z)^p} \qquad \textit{(induction hypothesis).} \end{aligned}$$

As a new application of Theorem 3.2.3, we will verify the fundamental property of the exponential.

Proposition 3.2.5 *If for any $z \in \mathbb{C}$, we define* $\exp(z)$ *as the series* $\sum \frac{z^n}{n!}$*, then:*

$$\forall a, b \in \mathbb{C}, \quad \exp(a+b) = \exp(a)\exp(b).$$

Proof: We can see $\exp(a)$ as the value at $z = 1$ of the series $\sum \frac{a^n}{n!} z^n$, and $\exp(b)$ as the value at $z = 1$ of the series $\sum \frac{b^n}{n!} z^n$. According to Theorem 3.2.3, the product of these two series is the series $\sum c_n z^n$, with:

$$\begin{aligned}
c_n &= \frac{a_n}{n!} + \frac{a^{n-1}}{(n-1)!}\frac{b}{1} + ... + \frac{a}{1}\frac{b^{n-1}}{(n-1)!} + \frac{b^n}{n!} \\
&= \sum_{k=0}^{n} \frac{a^k b^{n-k}}{k!(n-k)!} \\
&= \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} \\
&= \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\
&= \frac{(a+b)^n}{n!},
\end{aligned}$$

according to the binomial theorem.
The series $\sum c_n z^n$ therefore has sum $\exp((a+b)z)$, and its value at 1 is $\exp(a+b)$. □

In the previous proposition, we replaced z by az and bz in the exponential series. Replacing the variable z by a function of it is an operation that we frequently use. Here are two examples. For any z such that $|z| < 1$, we have:

$$1 + z + z^2 + ... + z^n + ... = \frac{1}{1-z}.$$

Replace z by $-z$:

$$1 - z + z^2 + ... + (-1)^n z^n + ... = \frac{1}{1+z}.$$

Replace z by z^2:

$$1 + z^2 + z^4 + ... + z^{2n} + ... = \frac{1}{1-z^2}.$$

We could have obtained this result in two other ways using Theorem 3.2.3, since:

$$\frac{1}{1-z^2} = \frac{\frac{1}{2}}{1-z} + \frac{\frac{1}{2}}{1+z} = \frac{1}{1-z}\frac{1}{1+z}.$$

We will now examine the properties of the sum of a power series, treated as a function of the variable z. In order not to complicate the definitions, we assume throughout this section that z is real. The identities obtained remain true for complex z, but it would be unnecessarily anticipating future chapters to leave the real domain.

3.3 Continuity of the Sum of a Power Series

3.3.1 Case of a Power Series of a Real Variable

Theorem 3.3.1 *The power series* $\sum_{n\in\mathbb{N}} a_n x^n$ *of the real variable x, with radius of convergence $R > 0$, is normally convergent on any* <u>segment</u> *included in the open interval of convergence* $]-R, R[$.

Proof: Let $[-r, r]$ with $r < R$ be a closed interval included in the open interval of convergence of the power series.

$$\forall\, x \in [-r, r] \quad |x| \leqslant r \quad \text{so} \quad \|a_n x^n\|_\infty^{[-r,r]} \leq |a_n|\, r^n.$$

Since $|a_n|\, r^n$ is the general term of a convergent series by definition of R, the series $\sum_{n\in\mathbb{N}} a_n x^n$ is normally convergent on $[-r, r]$. $\square$

Remark 3.3.2 *There is not necessarily normal convergence, or even uniform convergence, of the power series on* the entire *interval of convergence.*

Example 3.3.3 *The power series of the real variable* $\sum_{n\in\mathbb{N}^*} (-1)^{n-1} \frac{x^n}{n}$ *converges pointwise to* $x \mapsto \ln(1 + x)$ *on* $]-1, 1]$.
There is no normal convergence on $]-1, 1[$*, since* $\|u_n\|_\infty^{]-1,1[} = \frac{1}{n}\cdot$.
There is no uniform convergence on $]-1, 1[$*, because, otherwise, the double limit theorem would lead to a contradiction.*
However, there is uniform convergence on any segment $[a, 1]$ *with* $-1 < a < 1$*, by using the bound for the remainder of an alternating series.*

Theorem 3.3.4 *Let* $\sum_{n\in\mathbb{N}} a_n x^n$ *be a power series with radius of convergence* $R > 0$ $(x \in \mathbb{R})$*.*
Let S *denote the sum of the series* $\sum_{n\in\mathbb{N}} a_n x^n$*. Then* S *is continuous on each interval* $[-r, r] \subset\,]-R, R[$*.*

Proof: The functions $x \mapsto a_n x^n$ are continuous, and it is then sufficient to apply the theorem on the continuity of the sum of a series of functions. $\square$

Corollary 3.3.5 *Let* (a_n) *be a sequence of elements of* $\mathbb{C}$*, and* $\sum_{n\in\mathbb{N}} a_n x^n$ *be a power series of the real variable* x*, with radius of convergence* $R > 0$*. Let, for* $x \in]-R, R[$*,* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$*.*
Then, for any integer p*,* f *has, in a neighborhood of* 0*, the limited expansion:*

$$f(x) = \sum_{n=0}^{p} a_n x^n + O(x^{p+1}).$$

Proof: For any $x \in]-R, R[$ we have:

$$f(x) = \sum_{n=0}^{p} a_n x^n + \underbrace{x^{p+1} \left(\sum_{n=p+1}^{+\infty} a_n x^{n-p-1} \right)}_{=r_p(x)}.$$

Now, $\sum_{n=p+1}^{+\infty} a_n x^{n-p-1}$ is a power series with radius of convergence R; it is therefore, in particular, continuous at 0, hence bounded in a neighborhood of 0, that is, $r_p(x) = O(x^{p+1})$. $\square$

3.3.2 Continuity at the Boundary of the Interval of Convergence

Theorem 3.3.6 *Let $0 < R < +\infty$ be the radius of convergence of the series $\sum_{n\geq 0} a_n x^n$. Let $S = \sum_{n=0}^{+\infty} a_n x^n$ denote the sum of the series. If the series $\sum_{n\geq 0} a_n x^n$ converges at $x = R$ (resp. $x = -R$), then S is continuous at $x = R$ (resp. $x = -R$).*

Example 3.3.7 *Calculate the sum of the following series:*

$$\sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$$

To determine the radius of convergence, we apply the d'Alembert ratio test. Thus, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n}{n+1}\right| \underset{n\to+\infty}{\longrightarrow} .$$

Therefore, $R = 1 > 0$. Consequently, $\sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$ is absolutely convergent on $]-R, R[=]-1, 1[$. Moreover, $\sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$ is convergent at $x = 1 = R$ (since it is a Leibniz series), so by theorem 3.3.6, the sum $\sum_{n\geq 1}(-1)^{n-1}\frac{x^n}{n}$ is continuous at $x = 1$, i.e.,

$$\begin{aligned}\lim_{x\to 1}\left(\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^n}{n}\right) &= \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n} \\ &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} + \ldots \end{aligned} \tag{3.1}$$

Now, $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^n}{n}$ is the antiderivative (zero at $x = 0$) of the series

$$\sum_{n=0}^{+\infty}(-1)^n x^n = 1 - x + x^2 + \cdots + (-1)^n x^n + \ldots$$

and

$$\sum_{n=0}^{+\infty}(-1)^n x^n = \sum_{n=0}^{+\infty}(-x)^n = \frac{1}{1-(-x)}, \quad \forall x,\ |x| < 1$$

where

$$\sum_{n=0}^{+\infty}(-1)^n x^n = \frac{1}{1+x}, \quad \forall x,\ |x| < 1.$$

Therefore,

$$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{x^n}{n} = \int_0^x \frac{1}{1+t}\,dt = \ln(1+x) \tag{3.2}$$

Of (3.1) and (3.2), we deduce that

$$\lim_{x\to 1}\ln(1+x) = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} + \ldots$$

This is still equivalent to

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{n} + \ldots.$$

3.3.3 General Case

We have already defined the notion of continuity of a function defined on $\mathbb{R}$. In particular, saying that a map f defined on a subset D of $\mathbb{C}$ and with values in $\mathbb{C}$ is continuous at a point z_0 of D can be written as:

$$\forall\,\varepsilon > 0,\ \exists\,\alpha > 0 \text{ tel que } \forall\,z \in D,\ |z - z_0| < \alpha \Longrightarrow |f(z) - f(z_0)| < \varepsilon.$$

Some results on sequences and series of functions can also be extended without difficulty to functions from $\mathbb{C}$ to $\mathbb{C}$.

Definition 3.3.8 *Let (u_n) be a sequence of functions defined on a non-empty subset D of $\mathbb{C}$ and with values in $\mathbb{C}$. We say that this sequence converges pointwise on D if there exists a function $u\colon D \to \mathbb{C}$ such that: $\forall\,z \in D,\ \lim\limits_{n\to+\infty} u_n(z) = u(z)$.*

Definition 3.3.9 *Let (u_n) be a sequence of functions defined on a non-empty subset D of $\mathbb{C}$ and with values in $\mathbb{C}$.*

We say that this sequence converges uniformly on D if there exists a function $u\colon D \to \mathbb{C}$ such that $\lim\limits_{n\to+\infty} \|u_n - u\|_\infty^D = 0$.
We have denoted, as usual: $\|u_n - u\|_\infty^D = \sup\{|u_n(z) - u(z)| \mid z \in D\}$, and this definition assumes that the functions $u_n - u$ are bounded on D (at least from a certain rank).

Theorem 3.3.10 *If the u_n are continuous on D and if the sequence (u_n) converges uniformly on D to a function u, then u is continuous on D.*

Proof: The proof is exactly the same as for sequences of functions defined on an interval $I \subset \mathbb{R}$, simply replace the absolute value by the modulus. □

Definition 3.3.11 *Let $\sum\limits_{n\in\mathbb{N}} u_n$ be a series of functions defined on a subset D of $\mathbb{C}$ and with values in $\mathbb{C}$.*

1. *We say that this series converges pointwise on D if for any $z \in D$ the infinite series $\sum\limits_{n\in\mathbb{N}} u_n(z)$ is convergent.*
2. *We say that this series converges uniformly on D if it converges pointwise on D and if the sequence (R_n) of remainders converges uniformly on D to the null function.*
3. *We say that this series converges normally on D if the u_n are bounded on D (at least from a certain rank) and if the infinite series $\sum\limits_{n\in\mathbb{N}} \|u_n\|_\infty^D$ converges.*

We prove exactly in the same way as for series of functions of the real variable the following result:

Theorem 3.3.12 *If $(u_n)_{n\in\mathbb{N}}$ is a sequence of maps from $D \subset \mathbb{C}$ to $\mathbb{C}$ such that the series of functions $\sum\limits_{n\in\mathbb{N}} u_n$ is* normally convergent *on D, then:*

a) *For any $z \in D$, the series $\sum\limits_{n\in\mathbb{N}} u_n(z)$ is absolutely convergent in $\mathbb{C}$.*

b) *The series of functions $\sum\limits_{n\in\mathbb{N}} u_n$ is uniformly convergent on D.*

The following theorem generalizes Theorem 3.3.1 in the case of a power series of the complex variable.

Theorem 3.3.13 *Let (a_n) be a sequence of complex numbers. The power series $\sum_{n\in\mathbb{N}} a_n z^n$ of the complex variable z, with radius of convergence $R > 0$, is normally convergent on any* closed *disc included in the open disc of convergence.*

Proof: Let $r < R$ and $K = \{z \in \mathbb{C} \mid |z| \le r\}$ be the closed disc centered at 0 with radius r.

$$\forall\, z \in K, \quad |z| \leqslant r \quad \text{so} \quad \|a_n z^n\|_\infty^K \le |a_n|\, r^n.$$

Since $|a_n|\, r^n$ is the general term of a convergent series by definition of R, the series $\sum_{n\in\mathbb{N}} a_n z^n$ is normally convergent on K. □

Theorem 3.3.14 *Let (a_n) be a sequence of complex numbers and $\sum_{n\in\mathbb{N}} a_n z^n$ be a power series with radius of convergence $R > 0$ $(z \in \mathbb{C})$.*
The sum function $f\colon z \mapsto \sum_{n=0}^{+\infty} a_n z^n$ is continuous on its open disc of convergence.

Proof: The functions $z \mapsto a_n z^n$ are continuous. The convergence of the power series $\sum a_n z^n$ being normal, hence uniform, on any closed disc $K \subset \mathcal{B}(0,R)$, we deduce that f is continuous on any closed disc included in $\mathcal{B}(0,R)$, hence on $\mathcal{B}(0,R)$. □

Examples 3.3.15

1. *The power series $\sum_{n\in\mathbb{N}} z^n$ has radius of convergence 1 and :*

$$\forall\, z \in \mathcal{B}(0,1)\,,\ \sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}\,\cdot$$

It follows from the previous theorem that the function $z \mapsto \dfrac{1}{1-z}$ is continuous on $\mathcal{B}(0,1)$.

2. *The power series $\sum_{n\in\mathbb{N}} \dfrac{z^n}{n!}$ has radius of convergence $+\infty$ and :*

$$\forall\, z \in \mathbb{C},\ \sum_{n=0}^{+\infty} \frac{z^n}{n!} = \mathrm{e}^z.$$

It follows from the previous theorem that the function $z \mapsto \mathrm{e}^z$ is continuous on $\mathbb{C}$.

Remark 3.3.16 *In the complex case, the theorem 3.3.6 is formulated as follows:*
If $\sum_{n\ge 0} a_n z^n$ converges at z^0, a point on the circle of convergence of $\sum_{n\ge 0} a_n z^n$ $\big(z^0 \in \{z \in \mathbb{C} \mid |z| = R\}\big)$, then $S = \sum_{n=0}^{+\infty} a_n z^n$ is continuous at z^0.

3.4 Derivation. Integration

3.4.1 Derived Series

Definition 3.4.1 *Let* $\sum_{n\in\mathbb{N}} a_n z^n$ *be a power series.*
We call the *derived series* *of this series the power series:*

$$\left(\sum_{n\in\mathbb{N}} a_n z^n\right)' = \sum_{n\geq 1} n a_n z^{n-1} \quad = \left(\sum_{n\geq 0}(n+1)a_{n+1}z^n\right).$$

We call the *primitive series* *of this series the power series:*

$$\sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1} \qquad \left(= \sum_{n\geq 1} \frac{a_{n-1}}{n} z^n\right).$$

Theorem 3.4.2 *A power series, its derived series, and its primitive series have the same radius of convergence.*

Proof: It is sufficient to carry out the proof for the derived series (since the initial series is the derived series of its primitive series!).
Let R be the radius of convergence of the power series $\sum_{n\in\mathbb{N}} a_n z^n$, and R' that of the derived power series $\sum_{n\geq 1} n a_n z^{n-1}$.

- Suppose $R \neq +\infty$, and let z be such that $|z| > R$. Then the series $\sum_{n\in\mathbb{N}} |a_n z^n|$ is divergent.
  Since $\left|n a_n z^{n-1}\right| = \dfrac{n}{|z|} |a_n z^n| \geq |a_n z^n|$ for n large enough, the series $\sum_{n\geq 1} \left|n a_n z^{n-1}\right|$ also diverges, so $|z| > R'$.
  Thus, for any z, $|z| > R \Longrightarrow |z| > R'$; we deduce that $R \geq R'$, and this inequality remains true when $R = +\infty$.

- Suppose $R \neq 0$, and let z be such that $|z| < R$. Then there exists a real number r such that $|z| < r < R$. By definition of R, the sequence $(a_n r^n)$ is bounded: there exists a real number $M \geq 0$ such that, for any n we have: $|a_n| r^n \leq M$.
  We then have:
  $$\left|n a_n z^{n-1}\right| = n \frac{|a_n| r^n}{|z|} . \left|\frac{z}{r}\right|^n \leq n \frac{M}{|z|} . \left|\frac{z}{r}\right|^n$$
  so, since $\left|\dfrac{z}{r}\right| < 1$ and by comparison growth: $\lim_{n\to+\infty} n a_n z^{n-1} = 0$. This implies, according to one of the characterizations of the radius of convergence: $|z| \leq R'$.

  Thus, for any z, $|z| < R \Longrightarrow |z| \leq R'$; we deduce that $R \leq R'$, this inequality remaining true if $R = 0$.

- Finally, we have $R = R'$.

□

3.4.2 Derivation of a Power Series of a Real Variable

Theorem 3.4.3 *Let (a_n) be a sequence of elements of $\mathbb{C}$, and $\sum\limits_{n\in\mathbb{N}} a_n x^n$ be a power series of the real variable x, with radius of convergence $R > 0$. Let, for $x \in]-R, R[$, $f(x) = \sum\limits_{n=0}^{+\infty} a_n x^n$.*
Then f is of class C^1 on $]-R, R[$ and, for any $x \in]-R, R[$:

$$f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n \,.$$

Proof: It is sufficient to apply the theorem of term-by-term differentiation of a series of functions. Indeed, if we let $u_n : x \mapsto a_n x^n$:

- the u_n are indeed of class C^1 on $]-R, R[$;
- the series of functions $\sum\limits_{n\geq 0} u_n(x)$ converges pointwise to f on $]-R, R[$;
- according to the previous theorem, the derived series $\sum\limits_{n\geq 0} u_n'(x)$ has radius of convergence R, so according to Theorem 3.3.1, it converges uniformly on any segment included in $]-R, R[$: this is local uniform convergence.

The hypotheses of the theorem of term-by-term differentiation of a series of functions are therefore well verified, hence the result directly follows. □

Examples 3.4.4

1. *Calculation of $\sum\limits_{n=1}^{+\infty} n x^n$.*
The series $\sum\limits_{n\in\mathbb{N}} x^n$ has radius of convergence $R = 1$, and, for any $x \in]-1, 1[$, $f(x) = \sum\limits_{n=0}^{+\infty} x^n = \dfrac{1}{1-x}$. The previous theorem allows us to directly write, for any $x \in]-1, 1[$: $f'(x) = \sum\limits_{n=1}^{+\infty} n x^{n-1}$, so $\sum\limits_{n=1}^{+\infty} n x^{n-1} = \dfrac{1}{(1-x)^2}$ and finally: $\sum\limits_{n=1}^{+\infty} n x^n = \dfrac{x}{(1-x)^2}$.

2. *Calculation of $\sum\limits_{n=1}^{+\infty} n^2 x^n$.*
Using the same notation, and again using the previous theorem, we have, for any $x \in]-1, 1[$: $f''(x) = \sum\limits_{n=2}^{+\infty} n(n-1) x^{n-2}$, so $x^2 f''(x) = \sum\limits_{\substack{\cancel{n=2}\\ n=1}}^{+\infty} n(n-1) x^n$ then

$$\sum_{n=1}^{+\infty} n^2 x^n = x^2 f''(x) + \sum_{n=1}^{+\infty} n x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x+x^2}{(1-x)^3} \,.$$

Corollary 3.4.5 *With the same notation, the function f is of class C^∞ on $]-R, R[$ and, for any natural integer k and for any $x \in]-R, R[$:*

$$f^{(k)}(x) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = \sum_{n=0}^{+\infty} \frac{(n+k)!}{n!} a_{n+k} x^n \,.$$

Corollary 3.4.6 *With the same notation we have:*

$$\boxed{\forall\, k \in \mathbb{N}\ ,\ a_k = \frac{f^{(k)}(0)}{k!}\ .}$$

Corollary 3.4.7 *Let two power series be* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{+\infty} b_n x^n$.
If f *and* g *are defined and coincide in a neighborhood of* 0, *then* $a_n = b_n$ *for any integer* n.

Proof: Indeed, according to the previous theorem, we have, for any integer n, $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{g^{(n)}(0)}{n!} = b_n$. □

3.4.3 Integration of a Power Series of a Real Variable

Theorem 3.4.8 *Let* (a_n) *be a sequence of elements of* $\mathbb{C}$, *and* $\sum_{n\in\mathbb{N}} a_n x^n$ *be a power series of the real variable* x, *with radius of convergence* $R > 0$. *Let, for* $x \in]-R, R[$, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$.

a) For any segment $[a, b] \subset]-R, R[$, *we have;* $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \sum_{n=0}^{+\infty} \int_a^b (a_n x^n)\,\mathrm{d}x$.

b) In particular, for any $x \in]-R, R[$, *we have:* $\displaystyle\int_0^x f(t)\,\mathrm{d}t = \sum_{n=0}^{+\infty} \frac{a_n x^{n+1}}{n+1}$.

Proof: Since there is normal, and hence uniform, convergence of the series of functions on $[a, b]$ according to Theorem 3.3.1, we can apply the theorem of term-by-term integration of a series on a segment. □

Examples 3.4.9

1. *We know that, for* $|x| < 1$: $\dfrac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$.
   This is a power series with radius of convergence $R = 1$. *We therefore have, according to the previous theorem:*

$$\forall\, x \in]-1, 1[\ ,\ \int_0^x \frac{\mathrm{d}t}{1+t} = \sum_{n=0}^{+\infty} (-1)^n \int_0^x t^n\,\mathrm{d}t \quad \textit{that is} \quad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

   or again:

$$\forall\, x \in]-1, 1[\ ,\ \ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}.$$

   Using the alternating series test, we can show that the series of functions above converges uniformly on $[0, 1]$, *so the previous equality remains true for any* $x \in]-1, 1]$. *This was done in the chapter on series of functions.*

2. *We know that, for* $|x| < 1$: $\dfrac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$.

This is a power series with radius of convergence $R = 1$. We therefore have, according to the previous theorem:

$$\forall\, x \in]-1,1[\,,\ \int_0^x \frac{\mathrm{d}t}{1+t^2} = \sum_{n=0}^{+\infty} (-1)^n \int_0^x t^{2n}\,\mathrm{d}t$$

that is:

$$\forall\, x \in]-1,1[\ ,\ \arctan x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

This series of functions actually converges uniformly on the segment $[-1,1]$.

Indeed, it is easy to check that this series satisfies the hypotheses of the alternating series test on this segment, so it converges for any $x \in [-1,1]$ and, if we denote by $r_n(x)$ its remainder of order n, we will have:

$$\forall\, x \in]-1,1[\,,\ |r_n(x)| \leq \frac{|x|^{2n+3}}{2n+3}$$

so $\|r_n\|_\infty^{[-1,1]} \leq \dfrac{1}{2n+3}$ and $\lim\limits_{n\to+\infty} \|r_n\|_\infty^{[-1,1]} = 0$. This uniform convergence then implies the continuity of the sum function on the segment $[-1,1]$, so the equality above remains true for any $x \in [-1,1]$.

In particular, for $x = 1$ we (re)find the famous formula: $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$.

3.5 Power Series Expansions

Power series expansions extend the concept of polynomial approximations by expressing functions as infinite series. Here, we focus on functions defined in a neighborhood of 0 in the complex plane, which can be represented as the sum of a power series

3.5.1 Functions Expandable in Power Series

Definition 3.5.1

1. *We say that a function $f : \mathbb{C} \to \mathbb{C}$ is expandable in power series in a neighborhood V of 0, if there exists a power series $\displaystyle\sum_{n\in\mathbb{N}} a_n z^n$ with radius of convergence $R > 0$ such that, for any $z \in V$, $f(z) = \displaystyle\sum_{n=0}^{+\infty} a_n z^n$.*
2. *We say that a function $f : \mathbb{C} \to \mathbb{C}$ is expandable in power series in a neighborhood V of z_0, if there exists a power series with radius of convergence $R > 0$ such that, for any $z \in V$, $f(z) = \displaystyle\sum_{n=0}^{+\infty} a_n (z - z_0)^n$.*

Example 3.5.2 *The function $f : z \to f(z) = \dfrac{1}{1-z}$ is expandable in power series in the neighborhood V of 0 because for any z $(|z| < 1)$, we have*

$$\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n.$$

3.5.2 Taylor Series of a Function

For the rest of this section, we restrict ourselves to power series and functions of the real variable.

Definition 3.5.3 *Let f be of class C^∞ in a neighborhood V of x_0. We call the Taylor series of f in the neighborhood of x_0 the power series of the form*

$$\sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

If $x_0 = 0$, the series

$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n.$$

is called the Maclaurin series of f.

We can then state the following result.

Theorem 3.5.4 *Let f be a numerical function expandable in power series in a neighborhood of x_0, then the coefficients a_n of this series are of the form:*

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad \forall n \in \mathbb{N}.$$

The expansion of f in power series is unique and coincides with the sum of the Taylor series of f in a neighborhood of x_0, i.e.

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)(x_0)}}{n!}(x-x_0)^n \quad \forall x \ / \ |x-x_0| < R,$$

where $R > 0$ is the radius of convergence of the power series.

Proof: Since f is expandable in power series in a neighborhood of x_0, then we have for any $x \in]x_0 - R, x_0 + R[$

$$f(x) = \sum_{n=0}^{+\infty} a_n(x-x_0)^n. \tag{3.3}$$

Now f is infinitely differentiable on $]x_0 - R, x_0 + R[$, so

$$f^{(k)}(x) = \sum_{n=k}^{+\infty} a_n C_n^k k!(x-x_0)^{n-k}.$$

Hence,

$$f^{(k)}(x_0) = a_k C_k^k k! \Longrightarrow a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Thus, by replacing the value of a_n in (6.3.2), we obtain the Taylor series:

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \forall x \in |x-x_0| < R.$$

□

3.5.3 Functions Expandable in Taylor Series

Definition 3.5.5 *We say that a numerical function defined in a neighborhood of x_0 can be expanded in Taylor series in the neighborhood of x_0 if there exists $R > 0$ such that for any $x \in]x_0 - R, x_0 + R[$ we have*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Remark 3.5.6 *There exist functions of class C^∞ in a neighborhood of 0 whose Taylor series at 0 has a non-zero radius of convergence, but whose sum does not coincide with f.*

Example 3.5.7 *Let f be defined on $\mathbb{R}$ by:*

$$f(x) = \mathrm{e}^{-\frac{1}{x^2}} \text{ if } x \neq 0 \quad \text{et} \quad f(0) = 0\,.$$

We easily check, using the theorem of extension of class C^1 functions (iterated), that f is of class C^∞ on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for any n. Thus, the Taylor series of f gives the null function; this series has an infinite radius of convergence and does not coincide with f except at 0.
Let us systematically examine the definition and properties of the function f.
***Definition of the function** f.*

- *The function f is defined on $\mathbb{R}$ as:*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- *For $x \neq 0$, the function $f(x) = e^{-\frac{1}{x^2}}$ is highly regular: it is of class C^∞ (infinitely differentiable), with its derivatives decreasing rapidly as $x \to 0$.*
- *At $x = 0$, $f(0)$ is explicitly defined as 0.*

***Regularity of** f **on** $\mathbb{R}$*

- *Using the theorem on the extension of C^1 functions (iteratively applied), it can be shown that f is of class C^∞ on all of $\mathbb{R}$. This means that f is infinitely differentiable everywhere, including at $x = 0$.*

  The derivatives of f for $x \neq 0$ approach 0 as $x \to 0$. This property allows us to extend the definition of the derivatives at $x = 0$ by setting $f^{(n)}(0) = 0$ for all n.

***Taylor series of** f **near** 0*

- *The Taylor series of f in a neighborhood of 0 is expressed as:*

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

  Since $f^{(n)}(0) = 0$ for all n, the Taylor series is identically zero:

$$T_f(x) = 0 \quad \text{for all } x.$$

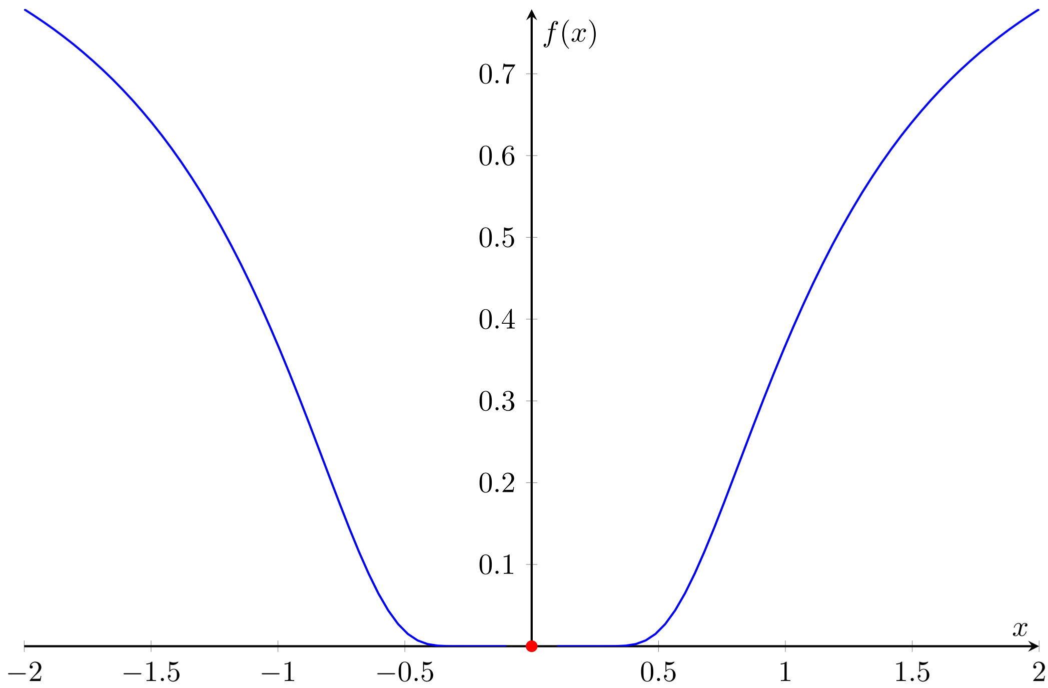
  Radius of convergence of the Taylor series The Taylor series of f has an infinite radius of convergence because all the coefficients $\dfrac{f^{(n)}(0)}{n!}$ are zero, ensuring perfect convergence for all $x \in \mathbb{R}$.

Comparison of f with its Taylor series

- *Although $T_f(x) = 0$ for all x, the function $f(x)$ is nonzero for $x \neq 0$. This illustrates an example where the Taylor series of a function does not match the function itself, except at the expansion point, $x = 0$.*

Conclusion
*This function f illustrates an important phenomenon in analysis: an infinitely differentiable function (C^∞) may have a Taylor series that completely diverges from the function, except at the expansion point. This is a classic example of a **flat function** at $x = 0$, where all derivatives vanish at that point.*

Graph of the function $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$



Remark 3.5.8 *There exist functions of class C^∞ in a neighborhood of 0 whose Taylor series at 0 has a zero radius of convergence.*

Example 3.5.9 *For any $x \in \mathbb{R}$, we let $f(x) = \sum_{n=0}^{+\infty} \dfrac{\cos(n^2 x)}{2^n}$.*
The results on the differentiation of series of functions allow us to easily show that f is of class C^∞ on $\mathbb{R}$. However, we can show that the radius of convergence of its Taylor series $\sum_{n=0}^{+\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ is zero.
Indeed:
By setting $u_n(x) = \dfrac{\cos(n^2 x)}{2^n}$, the u_n are of class C^∞ on $\mathbb{R}$ and, for any $k \in \mathbb{N}$ we have: $u_n^{(k)}(x) = n^{2k} \dfrac{\cos(n^2 x + k\frac{\pi}{2})}{2^n}$ so $\left\| u_n^{(k)} \right\|_\infty = \dfrac{n^{2k}}{2^n}$, which is the general term of a convergent series.
Thus, the series $\sum_{n \geq 0} u_n^{(k)}$ are normally, and therefore uniformly, convergent on $\mathbb{R}$, which proves that f is of class C^∞ on $\mathbb{R}$ and that: $\forall\, k \in \mathbb{N},\ \forall\, x \in \mathbb{R},\ f^{(k)}(x) = \sum_{n=0}^{+\infty} n^{2k} \dfrac{\cos(n^2 x + k\frac{\pi}{2})}{2^n}$.

Moreover, for $k \in \mathbb{N}$, $f^{(4k)}(0) = \sum\limits_{n=0}^{+\infty} \frac{n^{8k}}{2^n}$. *We therefore have, for any* $x \in \mathbb{R}$:

$$\left| f^{(4k)}(0) \frac{x^{4k}}{(4k)!} \right| = \sum_{n=0}^{+\infty} \frac{n^{8k}}{2^n} \frac{|x|^{4k}}{(4k)!} \geq \frac{(4k)^{8k}}{2^{4k}} \frac{|x|^{4k}}{(4k)!} \geq \frac{(4k)^{8k}}{2^{4k}} \frac{|x|^{4k}}{(4k)^{4k}} = k^{4k}(2|x|)^{4k}$$

by having bounded the sum by the term with $n = 4k$ *only and then by noting that* $(4k)! \leq (4k)^{4k}$, *so for any non-zero real number* x *we have* $\lim\limits_{k \to +\infty} \left| f^{(4k)}(0) \frac{x^{4k}}{(4k)!} \right| = +\infty$ *and the Taylor series of* f *diverges.*

Another method.

Let us consider the function $f(x)$ *defined by the series:*

$$f(x) = \sum_{n=0}^{+\infty} \frac{\cos(n^2 x)}{2^n}, \quad x \in \mathbb{R}.$$

Convergence of the Series

Each term of the series involves $\cos(n^2 x)$, *which is bounded for all* $x \in \mathbb{R}$, *i.e.,* $-1 \leq \cos(n^2 x) \leq 1$. *Additionally, the factor* $\frac{1}{2^n}$ *decays exponentially as* $n \to \infty$. *Therefore, for each* x, *the general term* $\frac{\cos(n^2 x)}{2^n}$ *is bounded by* $\frac{1}{2^n}$, *and since the series*

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

is a convergent geometric series, we conclude that the series for $f(x)$ *converges absolutely for all* $x \in \mathbb{R}$.

Thus, the function $f(x)$ *is well-defined for all* $x \in \mathbb{R}$.

Differentiability of $f(x)$

Now, let us check if $f(x)$ *is differentiable. The derivative of* $f(x)$ *is obtained by differentiating each term of the series:*

$$f'(x) = \sum_{n=1}^{\infty} \frac{-n^2 \sin(n^2 x)}{2^n}.$$

To determine if this series converges, we note that the terms are dominated by the exponential decay of $\frac{1}{2^n}$, *and the factor* n^2 *grows slower than* 2^n *as* $n \to \infty$. *Hence, the series for* $f'(x)$ *converges uniformly, and we conclude that* $f(x)$ *is differentiable for all* $x \in \mathbb{R}$.

Taylor Series at $x = 0$

Next, we consider the Taylor series of $f(x)$ *at* $x = 0$. *The Taylor series is given by:*

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

To find the Taylor series, we first compute the derivatives of $f(x)$ *at* $x = 0$. *Since each term of the series for* $f(x)$ *involves* $\cos(n^2 x)$, *we compute the value of the function and its derivatives at*

$x = 0$.
At $x = 0$, we have $\cos(n^2 \cdot 0) = 1$ for all n, so each term of the series at $x = 0$ is $\frac{1}{2^n}$. Therefore, the value of $f(x)$ at $x = 0$ is:

$$f(0) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Next, consider the derivatives at $x = 0$. Since $\sin(n^2 x)$ is zero when evaluated at $x = 0$, all derivatives of $f(x)$ at $x = 0$ are zero:

$$f^{(n)}(0) = 0 \quad \textit{for all} \quad n \geq 1.$$

Thus, the Taylor series of $f(x)$ at $x = 0$ is:

$$T_f(x) = f(0) + 0 \cdot x + 0 \cdot x^2 + \cdots = 2.$$

Radius of Convergence of the Taylor Series

The Taylor series at $x = 0$ is simply $T_f(x) = 2$, a constant function. This is very different from the original function $f(x)$, which is not constant and oscillates for $x \neq 0$.
The radius of convergence R of the Taylor series is determined by the distance from 0 to the nearest singularity of the function. Since the Taylor series does not represent the function at any point other than $x = 0$, the radius of convergence of the Taylor series is $R = 0$.

Conclusion

The function $f(x) = \sum_{n=0}^{\infty} \frac{\cos(n^2 x)}{2^n}$ is smooth (C^∞) and well-defined for all $x \in \mathbb{R}$. However, the Taylor series of $f(x)$ at $x = 0$ has a zero radius of convergence and is simply the constant 2. This is an example of a smooth function whose Taylor series at a point has zero radius of convergence, even though the function itself is smooth in a neighborhood of that point.

A necessary and sufficient condition for a function of class C^∞ on a neighborhood of 0 to be expandable in power series is given by the following theorem.

Theorem 3.5.10 *Let f be a function of class C^∞ on a neighborhood V of x_0. In order to have,*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \forall x \in V,$$

it is necessary and sufficient that

$$\lim_{n \to +\infty} r_n(x) = 0 \quad \forall x \in V,$$

where $r_n(x)$ is the remainder in Taylor's formula defined by:

$$r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \ \textit{with} \ \xi = x_0 + \theta(x - x_0), \ 0 < \theta < 1.$$

Proof: According to Taylor-Lagrange's formula, we have:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x) \quad \forall x \in V.$$

Which is still equivalent to

$$f(x) = S_n(x) + r_n(x) \quad \text{where} \quad S_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Hence,

$$\begin{aligned} & f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ \Longleftrightarrow & \lim_{n\to+\infty} S_n(x) = f(x) \Longleftrightarrow \lim_{n\to+\infty} r_n(x) = 0, \ \forall x \in V. \end{aligned}$$

□

The condition that the remainder tends to 0 when n tends to infinity is a priori very tedious to verify. However, there is a very simple sufficient condition for the expansion of f in Taylor series to be achieved. We then have,

Theorem 3.5.11 *We assume that:*

1. *f is of class C^∞ on a neighborhood $V =]x_0 - \delta, x_0 + \delta[$ ($\delta > 0$) of x_0.*
2. *$\exists M > 0$, $\forall x \in V$, we have*
$$\left|f^{(n)}(x)\right| \leq M \quad \forall n \in \mathbb{N}.$$

   Then, for $x \in V$, we have

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Proof: It is sufficient to show that

$$\lim_{n\to+\infty} r_n(x) = 0, \ \forall x \in V.$$

Indeed, we have

$$|r_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}\right| \leq M\frac{|x - x_0|^{n+1}}{(n+1)!} \leq M\frac{\delta^{n+1}}{(n+1)!}. \tag{3.4}$$

Now, the series $\displaystyle\sum_{n=0}^{+\infty} \frac{\delta^{n+1}}{(n+1)!}$ converges, so its general term tends to 0, i.e.

$$\lim_{n\to+\infty} \frac{\delta^{n+1}}{(n+1)!} = 0. \tag{3.5}$$

From (6.4.1) and (3.5), we deduce that $\lim_{n\to+\infty} r_n(x) = 0$. □

Remark 3.5.12 *The second condition of the previous theorem is sufficient for the expansion of f in Taylor series in a neighborhood of x_0 but it is not necessary. Indeed, we have:*

Example 3.5.13 *Let f be a function defined by:*

$$f(x) = \sum_{n\geq 0} \frac{2^n}{n!} x^n = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + ...$$

This series converges on $\mathbb{R}$ because

$$\lim_{n\to+\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0 < 1,$$

where $u_n = \dfrac{2^n}{n!}$.
Moreover, f is of class C^∞ on $\mathbb{R}$ and we have

$$\lim_{n\to+\infty} f^{(n)}(0) = \lim_{n\to+\infty} 2^n = +\infty.$$

Therefore,

$$\left| f^{(n)}(0) \right| \leq M \quad \forall x \in]-\delta, +\delta[,$$

is not verified.

3.5.4 Methods for Power Series Expansion

To expand a function in power series, we reduce it to expansions of standard functions. All these power series expansions are to be known by heart and are listed at the end of this chapter.

Using a Taylor Formula

Let f be a function of the real variable x, with values in $\mathbb{C}$, of class C^∞ in a neighborhood V of 0. We then have, for any $x \in V$ and any $n \in \mathbb{N}$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + r_n(x)\,.$$

To show that f is expandable in power series in a neighborhood of 0, it is therefore sufficient to show that there exists $r > 0$ such that, for any $x \in]-r, r[$ we have: $\lim\limits_{n\to\infty} r_n(x) = 0$.
To do this, we can use:

- Taylor-Lagrange's inequality: $|r_n(x)| \leq \dfrac{|x|^{n+1}}{(n+1)!} \sup\limits_{t\in[0,x]} \left| f^{n+1}(t) \right|$;
- or Taylor's formula with integral remainder: $r_n(x) = \displaystyle\int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t\,.$

Examples 3.5.14

1. ***Trigonometric functions :***
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the sine function. Then f is of class C^∞ on $\mathbb{R}$ and its derivative of order n exists and is given by:

$$f^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \quad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}.$$

Moreover, we have

$$\left| f^{(n)}(x) \right| \leq 1 \quad \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}.$$

We therefore conclude (cf. Theorem 3.5.11) that the sine function is equal to its Taylor series in a neighborhood of 0 on $\mathbb{R}$*:*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \forall x \in \mathbb{R}.$$

Which is still equivalent to

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} + ... \quad \forall x \in \mathbb{R}.$$

Similarly, we find the power series expansion of the cosine function, given by

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^{n+1} \frac{x^{2n}}{(2n)!} + ... \quad \forall x \in \mathbb{R}.$$

2. ***Exponential function :*** *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the exponential function. Then* f *is of class* C^∞ *on* $\mathbb{R}$ *and its derivative of order* n *exists and is given by:*

$$f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N}, \ \forall x \in \mathbb{R}.$$

Moreover, the Taylor-Lagrange remainder of this function is:

$$r_n(x) = \frac{e^c}{(n+1)!} x^{n+1} \quad \textit{where} \quad c = 0 + \theta(x - 0) = \theta x,$$

with $0 < \theta < 1$*. Since the series* $\displaystyle\sum_{n=0}^{+\infty} \frac{e^c}{(n+1)!} x^{n+1}$ *is convergent for any* $x \in \mathbb{R}$*, then its general term tends to 0, i.e.,*

$$\lim_{n \to +\infty} r_n(x) = \lim_{n \to +\infty} \frac{e^c}{(n+1)!} x^{n+1} = 0.$$

According to Theorem 3.5.10 (here the point $x_0 = 0$*), the power series expansion of the exponential function is given by*

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... \ \forall x \in \mathbb{R}.$$

Similarly, we find:

$$e^{-x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!} \ \forall x \in \mathbb{R}.$$

3. *Let* α *be a real number, and* $f(x) = (1+x)^\alpha$ *for* $x \in]-1, 1[$*.*

   Whatever the value of α*,* f *is of class* C^∞ *on* $\in]-1, 1[$ *and, for any* $k \in \mathbb{N}^*$ *we have:*

$$f^{(k)}(x) = \alpha(\alpha - 1) \dots (\alpha - k + 1)(1 + x)^{\alpha - k}.$$

   We will use here Taylor's formula with integral remainder:

$$\forall \, x \in \in]-1, 1[\ , \ f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + r_n(x) \quad \textit{with} \quad r_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, \mathrm{d}t$$

that is

$$r_n(x) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\int_0^x (x-t)^n(1+t)^{\alpha-n-1}\,\mathrm{d}t$$
$$= \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{\alpha-1}\,\mathrm{d}t.$$

Now, for t *between* 0 *and* x *(or* x *and* 0*),* $\left|\frac{x-t}{1+t}\right| \leq |x|$*. Indeed, the function* $t \mapsto \frac{x-t}{1+t}$ *is monotone on* $\in]-1,+\infty[$ *:*

- *The function* $t \mapsto \frac{x-t}{1+t}$ *is* ***monotonic*** *on the interval* $t \in]-1,+\infty[$*. This follows from the behavior of the numerator* $x-t$ *and the denominator* $1+t$*, which do not introduce discontinuities or oscillations in this range.*
- *Since the function is monotonic, the maximum value of* $\left|\frac{x-t}{1+t}\right|$ *on the interval* $[0,x]$ *(or* $[x,0]$*) is attained at one of the endpoints, namely* $t=0$ *or* $t=x$*.*
- *At* $t=0$*, we have*
$$\frac{x-t}{1+t} = \frac{x-0}{1+0} = x.$$
- *At* $t=x$*, we have*
$$\frac{x-t}{1+t} = \frac{x-x}{1+x} = 0.$$
- *Therefore,* $\left|\frac{x-t}{1+t}\right| \leq |x|$ *holds for all* t *in the given interval.*

So we have the inequality:

$$|r_n(x)| \leq \underbrace{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}x^n\right|}_{=u_n}\left|\int_0^x (1+t)^{\alpha-1}\,\mathrm{d}t\right|.$$

Now, the expression denoted u_n *above tends to* 0 *when* $n \to +\infty$*, since* $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\alpha-n-1}{n+1}x\right| \to |x| < 1$ *and by virtue of D'Alembert's rule.*

Therefore, for any $x \in \in]-1,1[$*,* $\lim_{n\to+\infty} r_n(x) = 0$ *and* $f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(0)}{k!}x^k$*, that is:*

$$\forall\, \alpha \in \mathbb{R}\ ,\ \forall\, x \in]-1,1[\ ,\ (1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$
$$+ \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \cdots$$

(if $\alpha \in \mathbb{N}$*, the sum above is finite and we retrieve the binomial formula).*

Using Integration or Differentiation

- We have already obtained, following Theorem 3.4.8, the power series expansions of the functions $x \mapsto \ln(1+x)$ and $x \mapsto \arctan x$ (cf. Example 3.4.9).

- We can obtain in the same way that of the function $x \mapsto \arcsin x$.

  Indeed, $f : x \mapsto \arcsin x$ is of class C^∞ on $]-1,1[$ and $f'(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$.

  According to the previous calculation $\alpha = -\dfrac{1}{2}$, for any $x \in]-1,1[$:

$$\begin{aligned}(1-x)^{-1/2} &= 1 + \sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} x^n = 1 + \sum_{n=1}^{+\infty} \frac{(2n)!}{2^n n!(2\cdot 4\cdots 2n)} x^n \\ &= 1 + \sum_{n=1}^{+\infty} \frac{(2n)!}{2^{2n}(n!)^2} x^n = \sum_{n=0}^{+\infty} \binom{2n}{n} \frac{x^n}{2^{2n}}\,.\end{aligned}$$

  Thus, for $t \in]-1,1[$, $\dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{n=0}^{+\infty} \binom{2n}{n} \frac{t^{2n}}{2^{2n}}$ and Theorem 3.4.8 allows us to directly conclude:

$$\forall\, x \in]-1,1[\ ,\ \arcsin x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \sum_{n=0}^{+\infty} \binom{2n}{n} \frac{x^{2n+1}}{2^{2n}(2n+1)}\,\cdot$$

- Another example: *determine the power series expansion in a neighborhood of the origin of the function* $f : x \mapsto \ln(1+x+x^2)$.

  We use Theorem 3.4.3 here. The function f is of class C^∞ on $\mathbb{R}$ and, for any real number x, we have: $f'(x) = \dfrac{2x+1}{1+x+x^2}$. We could obtain the power series expansion of this rational fraction by decomposing it into simple elements (see below), but a trick here allows us to simplify the calculations.

  We note that $f'(x) = \dfrac{(2x+1)(1-x)}{(1+x+x^2)(1-x)} = \dfrac{1+x-2x^2}{1-x^3}$; we will therefore have, for $|x| < 1$:

$$f'(x) = (1+x-2x^2)\sum_{n=0}^{+\infty} x^{3n} = \sum_{n=0}^{+\infty} x^{3n} + \sum_{n=0}^{+\infty} x^{3n+1} - 2\sum_{n=0}^{+\infty} x^{3n+2}$$

  or again $f'(x) = \sum\limits_{n=0}^{+\infty} a_n x^n$ with $a_n = 1$ if $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ and $a_n = -2$ if $n \equiv 2 \pmod 3$.

  By integrating, we then have, for $x \in]-1,1[$, $f(x) = f(0) + \displaystyle\int_0^x f'(t)\,\mathrm{d}t$, that is:

$$\forall\, x \in]-1,1[\ ,\ \ln(1+x+x^2) = \sum_{n=1}^{+\infty} a_{n-1} \frac{x^n}{n}\,\cdot$$

Linear Combination of Known Expansions

We use Theorem 3.2.1 here.

Examples 3.5.15

1. *From the expansion of* exp, *we easily obtain those of the functions* sh *and* ch, *using:*

$$\forall\, x \in \mathbb{R}\ ,\ \mathrm{ch}\, x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) \quad \textit{et} \quad \mathrm{sh}\, x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})\,\cdot$$

The series obtained, as for the function exp, *have a radius of convergence* ∞. *Then, we have*

$$\operatorname{ch} x = \sum_{n=1}^{+\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{(2n)!} + ... \quad \forall x \in \mathbb{R}.$$

And,

$$\operatorname{sh} x = \sum_{n=1}^{+\infty} \frac{x^{2n-1}}{(2n-1)!} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^{2n-1}}{(2n-1)!} + ... \quad \forall x \in \mathbb{R}.$$

2. Determine the power series expansion in a neighborhood of the origin of $f : x \mapsto \ln(1 + x - 2x^2)$.

We note that $1 + x - 2x^2 = (1-x)(1+2x)$, *so* f *is defined on* $I =]-\frac{1}{2}, 1[$, *and, for any* $x \in I$: $f(x) = \ln(1-x) + \ln(1+2x)$.

From the power series expansion of $x \mapsto \ln(1+x)$ *(which must be known by heart!), we deduce:*

$$\forall\, x \in]-1, 1[\ ,\ \ln(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n} \quad (\textit{Radius of convergence} = 1)$$

$$\forall\, x \in]-\frac{1}{2}, \frac{1}{2}[\ ,\ \ln(1+2x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(2x)^n}{n} \quad (\textit{Radius of convergence} = \frac{1}{2})$$

so:

$$\forall\, x \in]-\frac{1}{2}, \frac{1}{2}[\ ,\ f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}2^n - 1}{n} x^n \quad (\textit{Radius of convergence} = \frac{1}{2}).$$

3. Determine the power series expansion in a neighborhood of the origin of $f : x \mapsto \operatorname{ch} x \cos x$.

We write, for any real number x: $f(x) = \frac{1}{4}\left(\mathrm{e}^{(1+\mathrm{i})x} + \mathrm{e}^{(1-\mathrm{i})x} + \mathrm{e}^{(-1+\mathrm{i})x} + \mathrm{e}^{(-1-\mathrm{i})x}\right)$.

It is known that, for any $z \in \mathbb{C}$, $\mathrm{e}^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$, *we have, for any* $x \in \mathbb{R}$:

$$\mathrm{e}^{(1+\mathrm{i})x} = \sum_{n=0}^{+\infty} (1+\mathrm{i})^n \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^n \frac{x^n}{n!} = \sum_{n=0}^{+\infty} (\sqrt{2})^n \mathrm{e}^{\mathrm{i}n\frac{\pi}{4}} \frac{x^n}{n!}$$

and we obtain the power series expansion of $\mathrm{e}^{(1-\mathrm{i})x}$ *by taking the conjugate, hence*

$$\mathrm{e}^{(1+\mathrm{i})x} + \mathrm{e}^{(1-\mathrm{i})x} = \sum_{n=0}^{+\infty} (\sqrt{2})^n \left(\mathrm{e}^{\mathrm{i}n\frac{\pi}{4}} + \mathrm{e}^{-\mathrm{i}n\frac{\pi}{4}}\right) \frac{x^n}{n!} = 2\sum_{n=0}^{+\infty} (\sqrt{2})^n \cos\left(n\frac{\pi}{4}\right) \frac{x^n}{n!}.$$

Similarly:

$$\mathrm{e}^{(-1+\mathrm{i})x} + \mathrm{e}^{(-1-\mathrm{i})x} = 2\sum_{n=0}^{+\infty} (\sqrt{2})^n \cos\left(3n\frac{\pi}{4}\right) \frac{x^n}{n!}$$

hence

$$f(x) = \frac{1}{2}\sum_{n=0}^{+\infty} (\sqrt{2})^n \left(\cos\left(n\frac{\pi}{4}\right) + \cos\left(3n\frac{\pi}{4}\right)\right) \frac{x^n}{n!}.$$

But $\cos\left(n\frac{\pi}{4}\right) + \cos\left(3n\frac{\pi}{4}\right)$ *is zero unless* n *is a multiple of* 4, *and when* $n = 4p$, *this expression is equal to* $2\cos(p\pi) = 2(-1)^p$, *and finally:*

$$\forall\, x \in \mathbb{R}\ ,\ f(x) = \sum_{p=0}^{+\infty} (-1)^p \frac{2^{2p} x^{4p}}{(4p)!}.$$

Power Series Expansion of a Rational Fraction

Let $R(z) = \dfrac{P(z)}{Q(z)}$ be a rational fraction of the complex variable z *not admitting 0 as a pole* (i.e. $Q(0) \neq 0$). The decomposition into simple elements of R in $\mathbb{C}$ is written:

$$R(z) = \underbrace{E(z)}_{\text{integer part}} + \sum_i \sum_j \frac{\lambda_{i,j}}{(z-a_i)^j} \quad \text{(where the } a_i \text{ are the roots of } Q\text{)}$$

Now:

$$\frac{1}{(z-a_i)^j} = \frac{1}{(-a_i)^j} \cdot \frac{1}{\left(1 - \frac{z}{a_i}\right)^j} = \frac{1}{(-a_i)^j}\left(1 - \frac{z}{a_i}\right)^{-j}$$

(since $a_i \neq 0$ by assumption), and we saw on page 130 that:

$$\forall p \in \mathbb{N}^* \ , \ \forall z \in \mathbb{C} \text{ tq } |z| < 1 \ , \ \frac{1}{(1-z)^p} = \sum_{n=0}^{+\infty} \binom{n+p-1}{p-1} z^n$$

(we could also use the formula giving the power series expansion of $(1+x)^{-p}$).

We can therefore obtain the power series expansion of each simple element $\dfrac{1}{\left(1 - \frac{z}{a_i}\right)^j}$ for $\left|\dfrac{z}{a_i}\right| < 1$, that is, $|z| < |a_i|$, and the linear combination of the expansions thus obtained will give the power series expansion of the rational fraction R, with radius of convergence $R = \min(|a_i|)$.

Example 3.5.16 *Power series expansion, in a neighborhood of* 0*, of*

$$f : x \mapsto \frac{1-x^2}{x^2 - 2x\cos\theta + 1}$$

with $\theta \notin \pi\mathbb{Z}$.
f *is defined on* $\mathbb{R}$ *and the decomposition into simple elements is written, for any real number* x*:*

$$f(x) = \frac{1-x^2}{(x - \mathrm{e}^{\mathrm{i}\theta})(x - \mathrm{e}^{-\mathrm{i}\theta})} = -1 - \frac{\mathrm{e}^{\mathrm{i}\theta}}{x - \mathrm{e}^{\mathrm{i}\theta}} - \frac{\mathrm{e}^{-\mathrm{i}\theta}}{x - \mathrm{e}^{-\mathrm{i}\theta}}$$

(calculations to know how to do), that is:

$$f(x) = -1 + \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}\left(1 - \frac{x}{\mathrm{e}^{\mathrm{i}\theta}}\right)} + \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}\left(1 - \frac{x}{\mathrm{e}^{-\mathrm{i}\theta}}\right)} = -1 + \frac{1}{1 - x\mathrm{e}^{-\mathrm{i}\theta}} + \frac{1}{1 - x\mathrm{e}^{\mathrm{i}\theta}} \, .$$

For $\left|x\mathrm{e}^{\mathrm{i}\theta}\right| < 1$*, that is, for* $|x| < 1$*, we will therefore have:*

$$f(x) = -1 + \sum_{n=0}^{+\infty} (x\mathrm{e}^{\mathrm{i}\theta})^n + \sum_{n=0}^{+\infty} (x\mathrm{e}^{-\mathrm{i}\theta})^n = -1 + 2\sum_{n=0}^{+\infty} x^n \cos n\theta \, .$$

Using the Cauchy Product

This method, which relies on Theorem 3.2.3, generally leads to complicated expressions. Let's just give an example.

Example 3.5.17 Power series expansion, in a neighborhood of 0, of $f : x \mapsto \dfrac{\ln(1+x)}{1+x}$.
f is defined on $]-1,+\infty[$. *For* $x \in]-1,1[$, *we know the power series expansions of* $x \mapsto \ln(1+x)$ *and of* $x \mapsto \dfrac{1}{1+x}$, *both with radius of convergence equal to* 1. *Theorem 3.2.3 then allows us to assert that f will be expandable in a power series, and that the radius of convergence of the power series obtained will be* $R \geq 1$; *but since f is not defined at* -1, *we will actually have* $R = 1$.

We therefore have, for $|x| < 1$:

$$\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} = x \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n+1} \qquad et \qquad \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$$

(to apply the formula giving the Cauchy product of two power series, it is important that the two expansions start at $n = 0$!).

The Cauchy product series of the two series above is then written $\sum\limits_{n=0}^{+\infty} c_n x^n$ *with*

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \cdot (-1)^{n-k} = (-1)^n \sum_{k=0}^{n} \frac{1}{k+1}$$

and we will then have, for $x \in]-1,1[$: $f(x) = \sum\limits_{n=0}^{+\infty} c_n x^{n+1}$.

Differential Equation Method

We assume that the function f satisfies a certain differential equation and that f can be expanded as a power series in a neighborhood of 0, i.e., $f(x) = \sum\limits_{n=0}^{+\infty} a_n x^n$. Additionally, we know the power series expansions of the derivatives of f (Theorem 3.4.3), which allows us to substitute both f and its derivatives into the differential equation. This leads to a recurrence relation for the coefficients, which we can then solve.
The main issue with this approach (aside from the calculations) is that it assumes f can be expressed as a power series *a priori*. Therefore, we must verify a posteriori that the result is valid, meaning we must check that the radius of convergence of the obtained series is strictly positive.

Examples 3.5.18

1. *Let* $f(x) = (1+x)^\alpha$; *f is differentiable on* $]-1,1[$ *and* $f'(x) = \alpha(1+x)^{\alpha-1}$.

   The function f is therefore a solution of the linear differential equation $(1+x)y' = \alpha y$. *It is the unique solution of this equation on the interval* $]-1,1[$ *satisfying the initial condition* $f(0) = 1$ *(according to the Cauchy-Lipschitz theorem, since the function* $x \mapsto 1+x$ *is continuous and does not vanish on* $]-1,1[$).

   We then seek a power series $y(x) = \sum\limits_{n=0}^{+\infty} a_n x^n$, *with radius of convergence R strictly positive, satisfying the same differential equation and the same initial condition .*

   $y(0) = 1$ *is equivalent to* $a_0 = 1$ *and, since* $y'(x) = \sum\limits_{n=1}^{+\infty} n a_n x^{n-1}$ *for* $x \in]-R,R[$, *we have*

$$(1+x)y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} + \sum_{\substack{\cancel{n=1} \\ n=0}}^{+\infty} n a_n x^n = \alpha y(x) = \alpha \sum_{n=0}^{+\infty} a_n x^n$$

that is, after a change of index in the first sum:

$$\sum_{n=0}^{+\infty}\big[na_n+(n+1)a_{n+1}\big]x^n=\alpha\sum_{n=0}^{+\infty}a_nx^n.$$

We therefore have, by uniqueness of the power series expansion: $a_{n+1}=\dfrac{\alpha-n}{n+1}a_n$ *for all* $n\in\mathbb{N}$.

It is easy to deduce by recurrence, since $a_0=1$*:* $a_n=\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ *for* $n\in\mathbb{N}^*$.

Thus, $y(x)=1+\sum\limits_{n=1}^{+\infty}\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$.

Conversely, if we consider the function $g\colon x\mapsto 1+\sum\limits_{n=1}^{+\infty}\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$*, the D'Alembert rule allows us to easily prove that this power series has radius of convergence* $R=1$*, therefore is of class* C^1 *on* $]-1,1[$.
By then repeating the calculations above in the other direction, we obtain that g *is indeed a solution of the differential equation on* $]-1,1[$*. It therefore coincides with* f *on* $]-1,1[$ *(by uniqueness of the solution to the Cauchy problem), which allows us to find the known expansion of* $x\mapsto(1+x)^\alpha$.

2. Determine the power series expansion in a neighborhood of 0 of the function f defined on $\mathbb{R}$ by: $f(x)=\mathrm{e}^{x^2}\displaystyle\int_0^x\mathrm{e}^{-t^2}\,\mathrm{d}t$.

The function $x\mapsto\mathrm{e}^{-x^2}$ *is expandable in a power series since, for all* $x\in\mathbb{R}$ *we have* $\mathrm{e}^{-x^2}=\sum\limits_{n=0}^{+\infty}\dfrac{(-x^2)^n}{n!}$*. By integration, it follows that the function* $x\mapsto\displaystyle\int_0^x\mathrm{e}^{-t^2}\,\mathrm{d}t$ *is also expandable in a power series. We also know the power series expansion of* $x\mapsto\mathrm{e}^{x^2}$*, and the two series have radius of convergence* $+\infty$.
It would be very awkward to write the Cauchy product series, but Theorem 3.5.11 nevertheless allows us to assert that f *is expandable in a power series, and that the series obtained will have radius of convergence* $+\infty$*. Moreover,* f *is odd, so ultimately there exists a sequence* (a_n) *such that* $f(x)=\sum\limits_{n=0}^{+\infty}a_nx^{2n+1}$ *for all real numbers* x.

We then notice that, for all real numbers x, $\mathrm{e}^{-x^2}f(x)=\displaystyle\int_0^x\mathrm{e}^{-t^2}\,\mathrm{d}t$ *and, by differentiating this relationship, we obtain* $f'(x)-2xf(x)=1$.

By differentiation of a power series, we have $f'(x)=\sum\limits_{n=0}^{+\infty}(2n+1)a_nx^{2n}$ *so by replacing in the previous equality:*

$$\forall x\in\mathbb{R},\ \sum_{n=0}^{+\infty}(2n+1)a_nx^{2n}-2\sum_{n=0}^{+\infty}a_nx^{2n+2}=1$$

and after a change of index in the 2nd sum:

$$\sum_{n=0}^{+\infty}(2n+1)a_nx^{2n}-2\sum_{n=1}^{+\infty}a_{n-1}x^{2n}=1\quad\text{that is:}\quad a_0+\sum_{n=1}^{+\infty}\big[(2n+1)a_n-2a_{n-1}\big]x^{2n}=1.$$

By uniqueness of the power series expansion, we obtain: $a_0=1$ *and, for all* $n\in\mathbb{N}^*$*:* $a_n=\dfrac{2}{2n+1}a_{n-1}$.

We deduce, for $n \in \mathbb{N}^*$:

$$a_n = \frac{2}{2n+1}a_{n-1} = \frac{2}{2n+1}\cdot\frac{2}{2n-1}a_{n-2} = \cdots = \frac{2}{2n+1}\cdot\frac{2}{2n-1}\cdot\frac{2}{2n-3}\cdots\frac{2}{3}a_0$$

or:

$$a_n = \frac{2^n}{(2n+1)(2n-1)\cdots 3} = \frac{2^n(2n)(2n-2)\cdots 2}{(2n+1)!} = \frac{2^{2n}n!}{(2n+1)!}$$

this equality being still true for $n = 0$.

Finally: $\forall\, x \in \mathbb{R}\ ,\ f(x) = \sum\limits_{n=0}^{+\infty} \frac{2^{2n}n!}{(2n+1)!}x^{2n+1}$.

3. Determine the power series expansion in a neighborhood of 0 of the function f defined on $[-1,1]$ by: $f(x) = (\arcsin x)^2$.
We know that the function $x \mapsto \arcsin x$ is expandable in a power series on $]-1,1[$ (by integration of the power series expansion of $x \mapsto (1-x^2)^{-\frac{1}{2}}$, see page 43).

Theorem 3.5.11 then allows us to assert that f is expandable in a power series, and that the radius of convergence of the power series obtained will be $R \geq 1$; but since f is not defined for $x > 1$, we will actually have $R = 1$.
f is of class C^∞ on $]-1,1[$ as a product of such functions, and " some " calculations of derivatives (not reproduced), show that, for any

$$x \in]-1,1[\ :\ (1-x^2)f''(x) - xf'(x) = 2.$$

According to the Cauchy-Lipschitz theorem, f' is then the only solution on $]-1,1[$ of the differential equation $(1-x^2)y' - xy = 2$ which vanishes at 0.
Since f is even, f' is odd, and as we have seen that it is expandable in a power series on $]-1,1[$, there exist real numbers a_n such that, for any

$$x \in]-1,1[\ :\ f'(x) = \sum_{n=0}^{+\infty} a_n x^{2n+1}.$$

By differentiation of a power series, we have

$$f''(x) = \sum_{n=0}^{+\infty}(2n+1)a_n x^{2n},$$

and by substituting into the differential equation we obtain:

$$(1-x^2)\sum_{n=0}^{+\infty}(2n+1)a_n x^{2n} - x\sum_{n=0}^{+\infty} a_n x^{2n+1} = 2$$

$$\sum_{n=0}^{+\infty}(2n+1)a_n x^{2n} - \sum_{n=0}^{+\infty}(2n+1)a_n x^{2n+2} - \sum_{n=0}^{+\infty} a_n x^{2n+2} = 2$$

$$\sum_{n=0}^{+\infty}(2n+1)a_n x^{2n} - \sum_{n=0}^{+\infty}(2n+2)a_n x^{2n+2} = 2$$

and finally:

$$\sum_{n=0}^{+\infty}(2n+1)a_n x^{2n} - \sum_{n=1}^{+\infty} 2na_{n-1}x^{2n} = 2 \text{ (change of index } n' = n+1 \text{ in the 2nd sum).}$$

By uniqueness of the power series expansion, we deduce: $a_0 = 2$ *and, for all* $n \geq 1$*:* $a_n = \dfrac{2n}{2n+1} a_{n-1}$ *.*

Hence, for all $n \in \mathbb{N}^*$*:*

$$\begin{aligned} a_n &= \frac{2n}{2n+1} a_{n-1} = \frac{2n}{2n+1} \frac{2n-2}{2n-1} a_{n-2} = \cdots = \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} a_1 \\ &= 2\, \frac{\big[(2n)(2n-2)\cdots 2\big]^2}{(2n+1)!} = 2\, \frac{[2^n n!]^2}{(2n+1)!} \, . \end{aligned}$$

Thus: $\forall\, x \in]-1,1[\ ,\ f'(x) = \sum\limits_{n=0}^{+\infty} \dfrac{2^{2n+1}(n!)^2}{(2n+1)!} x^{2n+1}$ *and we deduce* f *by integration (using* $f(0) = 0$*).*

3.6 Summation of a Power Series

Here we are dealing with the inverse problem of the previous one: given a power series $\sum\limits_{n \in \mathbb{N}} a_n x^n$, with radius of convergence $R > 0$, it is a matter of expressing its sum using the usual functions. There is no general method to do this, but we can try to exploit the following leads:

- Directly make known power series expansions appear, in $\sum a_n x^n$. To do this, we can (possibly): decompose a_n as a linear combination of simpler terms, use a change of variable, use derivation or integration, recognize a Cauchy product...
- Using a recurrence relation between the a_n, determine a differential equation whose sum function is a solution.

Examples 3.6.1

1. Determine the radius of convergence and the sum of the power series $\sum\limits_{n=0}^{+\infty} \dfrac{x^{2n+1}}{2n+1}$.

The sequence $\left(\dfrac{x^{2n+1}}{2n+1}\right)_{n \in \mathbb{N}}$ *is bounded if and only if* $|x| \leq 1$*, so* $R = 1$*.*

a) 1st solution:

For $x \in]-1,1[$*, let* $f(x) = \sum\limits_{n=0}^{+\infty} \dfrac{x^{2n+1}}{2n+1}$ ·*. By differentiation of a power series,* $f'(x) = \sum\limits_{n=0}^{+\infty} x^{2n} = \dfrac{1}{1-x^2}$ *(geometric series with ratio* x^2*). It only remains to integrate, using* $f(0) = 0$.

b) 2nd solution:

We directly use the power series expansions from the course:

$$\forall\, x \in]-1,1[,\ \ln(1+x) = \sum_{n=1}^{+\infty} (-1)^n \frac{x^n}{n} \qquad et \qquad \ln(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n} \, .$$

By one of these methods we find, for all $x \in]-1,1[\ :\ f(x) = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ *.*

2. Determine the radius of convergence and the sum of the power series $\sum\limits_{n=0}^{+\infty} \dfrac{n x^n}{(2n+1)!}$.

By comparing growth of power functions and factorials, we have $\lim\limits_{n\to+\infty} \dfrac{nx^n}{(2n+1)!} = 0$*; in particular, for all* $x \in \mathbb{R}$*, the sequence* $\left(\dfrac{nx^n}{(2n+1)!}\right)$ *is bounded, so the radius of convergence is* $R = +\infty$.
By writing : $\dfrac{nx^n}{(2n+1)!} = \dfrac{1}{2}\left(\dfrac{x^n}{(2n)!} - \dfrac{x^n}{(2n+1)!}\right)$*, then by setting* $t = \sqrt{x}$ *if* $x > 0$ *or* $t = \sqrt{-x}$ *if* $x < 0$*, we are reduced to known power series, and we obtain:*

$$f(x) = \begin{cases} \dfrac{1}{2}\left(\cos(\sqrt{-x}) - \dfrac{\sin\sqrt{-x}}{\sqrt{-x}}\right) & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ \dfrac{1}{2}\left(\mathrm{ch}(\sqrt{x}) - \dfrac{\mathrm{sh}\,\sqrt{x}}{\sqrt{x}}\right) & \text{for } x > 0 \end{cases} .$$

3. Determine the radius of convergence and the sum of the power series $\displaystyle\sum_{n=0}^{+\infty} \frac{(2n+1)!}{(n!)^2} x^{2n}$.

By setting $a_n = \dfrac{(2n+1)!}{(n!)^2}$*, we have the recurrence relation*

$$(n+1)a_{n+1} = 2(2n+3)a_n$$

a) This allows us to show (by D'Alembert) that the radius of convergence is $R = \dfrac{1}{2}\cdot$

b) Let f *be the sum of the power series. Then, by the theorem of differentiation of a power series we have:*

$$\forall\, x \in]-\frac{1}{2}, \frac{1}{2}[,\ f(x) = \sum_{n=0}^{+\infty} a_n x^{2n}$$

and

$$f'(x) = \sum_{n=1}^{+\infty} 2na_n x^{2n-1} = \sum_{n=0}^{+\infty} 2(n+1)a_{n+1}x^{2n+1}.$$

Taking into account the recurrence relation cited above, we therefore have, for $x \in]-\frac{1}{2}, \frac{1}{2}[$ *:*

$$\begin{aligned} f'(x) &= 2\sum_{n=0}^{+\infty}(n+1)a_{n+1}x^{2n+1} \\ &= 4\sum_{n=0}^{+\infty}(2n+3)a_n x^{2n+1} \\ &= 4x^2 \sum_{\substack{\cancel{n=0} \\ n=1}}^{+\infty} 2na_n x^{2n-1} + 12x\sum_{n=0}^{+\infty} a_n x^{2n}. \end{aligned}$$

f *is therefore a solution on* $]-\frac{1}{2}, \frac{1}{2}[$ *of the differential equation :*

$$(1-4x^2)f'(x) = 12xf(x)\,.$$

This is a linear homogeneous differential equation of the 1st order, whose general solution on $]-\frac{1}{2}, \frac{1}{2}[$ *(interval where the coefficient of* $f'(x)$ *does not vanish) is given by*

$$f(x) = C\mathrm{e}^{\int \frac{12x}{1-4x^2}\,\mathrm{d}x}.$$

Taking into account $f(0) = 1$*, we find* $f(x) = (1-4x^2)^{-3/2}$ *for* $x \in]-\dfrac{1}{2}, \dfrac{1}{2}[$.

4. Determine the radius of convergence and the sum of the power series $\sum_{n=1}^{+\infty} \frac{x^n}{n\binom{2n}{n}}$.

By setting $a_n = \frac{1}{n\binom{2n}{n}}$ *for* $n \geq 1$ *we easily find the recurrence relation* $a_{n+1} = \frac{n}{2(2n+1)} a_n$ *for* $n \geq 1$.

a) This allows us to show (by D'Alembert) that the radius of convergence is $R = 4$.

b) Let f *be the sum of the power series. Then, by the theorem of differentiation of a power series we have:*

$$\forall\, x \in]-4, 4[,\ f(x) = \sum_{n=1}^{+\infty} a_n x^n \quad et \quad f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$$

Taking into account the recurrence relation cited above, we therefore have, for $x \in]-4, 4[$ *and* $x \neq 0$ *:*

$$\begin{aligned} 4f'(x) &= 4a_1 + \sum_{n=1}^{+\infty} 4(n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} 2(2n+1) a_{n+1} x^n + 2 \sum_{n=1}^{+\infty} a_{n+1} x^n \\ &= 4a_1 + \sum_{n=1}^{+\infty} n a_n x^n + 2 \sum_{n=2}^{+\infty} a_n x^{n-1} \\ &= 4a_1 + x f'(x) + \frac{2}{x} \sum_{n=2}^{+\infty} a_n x^n = 4a_1 + x f'(x) + \frac{2}{x}\left(f(x) - a_1 x\right) \end{aligned}$$

and taking into account $a_1 = \frac{1}{2}$ *we find (a priori for* $x \neq 0$ *but the equality remains true for* $x = 0$ *by continuity):*

$$\forall\, x \in]-4, 4[,\ x(x-4) f'(x) + 2f(x) = -x\,.$$

It only remains to solve this linear differential equation of the 1st order, non homogeneous. Taking into account the coefficient of $f'(x)$*, we must distinguish the cases* $x > 0$ *and* $x < 0$.

- 1st case: $x \in]0, 4[$:

  The associated homogeneous equation then is written: $f'(x) = \frac{2}{x(4-x)} f(x)$ *and has for general solution the functions of the form*

$$x \mapsto C e^{\int \frac{2}{x(4-x)}\, dx}.$$

  We write the decomposition into simple elements:

$$\frac{2}{x(4-x)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{4-x}\right)$$

  so

$$e^{\int \frac{2}{x(4-x)}\, dx} = e^{\frac{1}{2}(\ln x - \ln(4-x))} = C\sqrt{\frac{x}{4-x}}\,.$$

  By the method of variation of constants, we then seek $f(x)$ *in the form* $f(x) = C(x)\sqrt{\frac{x}{4-x}}$ *which leads to*

$$x(x-4) C'(x) \sqrt{\frac{x}{4-x}} = -x$$

that is
$$C'(x) = \frac{1}{\sqrt{x(4-x)}}\cdot$$
By usual methods (putting the trinomial into canonical form):
$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x(4-x)}} &= \int \frac{\mathrm{d}x}{\sqrt{-(x-2)^2+4}}\\ &= \int \frac{\mathrm{d}x}{2\sqrt{1-\left(\frac{x-2}{2}\right)^2}} = \arcsin\left(\frac{x-2}{2}\right) + cste\cdot\end{aligned}$$
We therefore arrive at
$$f(x) = \left(\arcsin\left(\frac{x-2}{2}\right) + cste\right)\sqrt{\frac{x}{4-x}}$$
Unfortunately, the only relation $f(0)=0$ is not enough to determine the constant. But the expression of f as a power series gives $f'(0) = a_1 = \frac{1}{2}$, so by definition of the derivative, since $f(0)=0$, we have $\lim_{x\to 0}\frac{f(x)}{x} = \frac{1}{2}$. In particular, the limit at 0 of $\frac{f(x)}{x}$ must be finite, which requires $\lim_{x\to 0^+}\arcsin\left(\frac{x-2}{2}\right) + cste = 0$, that is, $cste = \frac{\pi}{2}$. And taking into account the well-known relation $\arcsin y + \arccos y = \frac{\pi}{2}$ for $y\in[-1,1]$ we finally have:
$$\forall\, x \in]0,4[\ ,\ f(x) = \arccos\left(\frac{2-x}{2}\right)\sqrt{\frac{x}{4-x}}\,.$$

- 2nd case: $x\in]-4,0[$:
  It is the same principle, but here it will appear $\sqrt{x(x-4)}$ and we will use the primitive $\int \frac{\mathrm{d}t}{\sqrt{1+t^2}} = \ln\left(t+\sqrt{1+t^2}\right)$...

5. Determine the radius of convergence and the sum of the power series $\sum_{n=0}^{+\infty}(-1)^n\frac{\binom{2n}{n}}{2n-1}x^n$.

Here again, the important thing is to start by determining a recurrence relation between a_n and a_{n+1}.

We find here $R=\frac{1}{4}$; f is a solution of the differential equation: $(4x+1)f'(x) = 2f(x)$.

Taking into account $f(0)=-1$, we find: $f(x) = -\sqrt{4x+1}$ for $x\in]-\frac{1}{4},\frac{1}{4}[$.

3.6.1 Complex Trigonometric Functions

Let $z\in\mathbb{C}$, we represent by $\sin z$ (resp. $\cos z$), the sum of the series $\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$ (resp. $\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}z^{2n}$) i.e., for all $z\in\mathbb{C}$
$$\sin z = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\ \text{(resp. } \cos z = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}z^{2n}\text{)}.$$

The mapping $z \longrightarrow \sin z$ (resp. $z \longrightarrow \cos z$) defines the sine (resp. cosine) function of the complex variable. The following lemmas allow us to recover the properties of the exponential function, as well as those of the usual trigonometric functions.

Lemma 3.6.2 *For all $z \in \mathbb{C}$, we have*

$$e^{iz} = \cos z + i \sin z.$$

Proof: Indeed, we have for all $z \in \mathbb{C}$

$$e^{iz} = \sum_{n=0}^{+\infty} \frac{(iz)^n}{n!} = \sum_{k=0}^{+\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^{+\infty} \frac{(iz)^{2k+1}}{(2k+)!}$$

Thus,

$$e^{iz} = \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{(2k)!} + i \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k+1}}{(2k+)!}.$$

Consequently,

$$e^{iz} = \cos z + i \sin z.$$

□ As a consequence, we have

Lemma 3.6.3 *For all $z \in \mathbb{C}$, we have*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \textit{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Moreover,

$$\cos^2 z + \sin^2 z = 1.$$

3.6.2 Complex Hyperbolic Functions

Let $z \in \mathbb{C}$, we represent by $\operatorname{sh} z$ (resp. $\operatorname{ch} z$) the sum of the series $\sum_{n \geq 0} \frac{z^{2n+1}}{(2n+1)!}$ (resp. $\sum_{n \geq 0} \frac{z^{2n}}{(2n)!}$) i.e.,

$$\operatorname{sh} z = \sum_{n \geq 0} \frac{z^{2n+1}}{(2n+1)!} \quad (\text{resp. } \operatorname{ch} z = \sum_{n \geq 0} \frac{z^{2n}}{(2n)!}).$$

The mapping $z \longrightarrow \operatorname{sh} z$ (resp. $z \longrightarrow \operatorname{ch} z$) defines the hyperbolic sine (resp. hyperbolic cosine) function of the complex variable.

Lemma 3.6.4 *For all $z \in \mathbb{C}$, we have*

$$\operatorname{sh} z = \frac{e^z - e^{-z}}{2} \quad \textit{and} \quad \operatorname{ch} z = \frac{e^z + e^{-z}}{2}.$$

Proof: The proof follows from the definition of e^z and e^{-z}. □

3.6.3 Complex Logarithm

It is natural to complete the definition of the complex exponential by that of the logarithm, as the inverse function of the previous one. This function must of course coincide with its real counterpart.

Definition 3.6.5 *We call logarithm of a complex number $z \in \mathbb{C}$, any complex number Z, such that*

$$e^Z = z.$$

That is

$$\log z = Z = \ln|z| + i \arg z.$$

The mapping $z \longrightarrow \log z$ defines the logarithm function of the complex variable.

Remark 3.6.6 *We will note that, since the argument of the complex number z is only defined modulo 2π, the logarithm function will only be a "true function" if we further impose a condition on the values taken by the argument of z. More precisely, if we set $z = |z|(\cos\theta + i\sin\theta)$ where $\theta := \arg z$ and $Z = X + iY$ then $X = \ln|z|$ and $Y = \theta + 2k\pi$, $k \in \mathbb{Z}$. Indeed, by definition of the logarithm we have*

$$e^Z = z.$$

This is equivalent, by replacing Z and z by their respective values, to

$$e^X(\cos Y + i\sin Y) = |z|(\cos\theta + i\sin\theta).$$

From this equality, we obtain

$$e^X = |z| \text{ and } Y = \theta + 2k\pi, \quad k \in \mathbb{Z}.$$

Thus, the logarithm of a complex number z is given by the formula:

$$\log z = Z = X + iY = \ln|z| + i(\theta + 2k\pi), \quad k \in \mathbb{Z}.$$

If we impose conditions on θ, we define thus a complex logarithm function, denoted log, *defined by*

$$\log z = \ln|z| + i \arg z.$$

Remark 3.6.7 *We can prove that for all z such that $|z| < 1$,*

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + ... + (-1)^{n-1}\frac{z^n}{n} + ...$$

3.7 Common Power Series Expansions

A few remarks about the following table:

- The power series expansions of $\sqrt{1+x}$ and $\dfrac{1}{\sqrt{1+x}}$ are of course not to be memorized by heart; they are simple applications of the expansion of $(1+x)^\alpha$ for $\alpha = \pm\dfrac{1}{2}$.

- Similarly, the power series expansion of arcsin is simply obtained by integrating the expansion of $\dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$.

Function	Power Series Expansion	Radius of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$+\infty$
$\log(1+x)$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	1
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	1
$(1+x)^\alpha$	$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$	1
$\sqrt{1+x}$	$\sum_{n=0}^{\infty} \binom{1/2}{n} x^n$	1
$\frac{1}{\sqrt{1+x}}$	$\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$	1
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$+\infty$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$+\infty$
$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$+\infty$
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$+\infty$
$\arcsin x$	$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}$	1
$\arctan x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	1

3.8 Exercises of the Chapter

Exercise 3.8.1 *Determine the radius of convergence of the power series $\sum a_n z^n$ in each of the following cases:* ***a)*** $a_n = \sqrt[n]{n}$ ***b)*** $a_n = \frac{n^n}{n!}$ ***c)*** $a_n = \frac{\sin n}{n}$ ***d)*** $a_n = \arctan\left(\frac{1}{n^\alpha}\right)$ $(\alpha \in \mathbb{R})$
e) $a_n = \binom{2n}{n}$ ***f)*** $\sqrt[n+1]{n+1} - \sqrt[n]{n}$ ***g)*** $a_n = \tan\left(\frac{n\pi}{7}\right)$ ***h)*** $a_n =$ *sum of the divisors of n.*

Correction 3.8.1

a) $\lim_{n\to+\infty} a_n = 1$ *so the sequence $(a_n z^n)$ is bounded if and only if $|z| \leq 1$.*

By definition, we deduce $R = 1$.

b) *Let* $z \in \mathbb{C}^*$. $\dfrac{\left|a_{n+1}z^{n+1}\right|}{|a_n z^n|} = \left(\dfrac{n+1}{n}\right)^n |z| \underset{n\to+\infty}{\longrightarrow} \mathrm{e}\,|z|$ *so, according to d'Alembert's rule for series with positive real terms, we have:*

- *if* $\mathrm{e}\,|z| < 1$ *the series* $\sum |a_n z^n|$ *converges;*
- *if* $\mathrm{e}\,|z| > 1$ *this series diverges.*

Therefore, $R = \dfrac{1}{\mathrm{e}}\cdot$

c) *If* $|z| \leq 1$ *the sequence* $(a_n z^n)$ *is bounded so* $R \geq 1$.

Let z *be such that* $|z| > 1$. *We can always find integers* n *as large as we want such that* $|\sin n| \geq \frac{1}{2}$ *(it suffices to take* $n \in \left[\frac{\pi}{6} + k\pi, \frac{5\pi}{6} + k\pi\right]$ *with* $k \in \mathbb{N}$*, which is always possible since the length of the interval* $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ *is greater than* 1*). For these values of* n*, we have* $|a_n z^n| \geq \dfrac{|z|^n}{2n}$*; hence, by comparing growth rates, the sequence* $(a_n z^n)$ *is unbounded.*

We deduce $R \leq 1$ *and finally,* $R = 1$.

Another solution; *the radius of convergence of the series* $\sum \dfrac{\sin n}{n} z^n$ *is the same as that of its derived series, and thus the same as that of the series* $\sum \sin n z^n$.

However, we know that the sequence $(\sin n)$ *diverges (classic exercise, already done), so the series diverges for* $z = 1$ *hence* $R \leq 1$.

d)
- *If* $\alpha < 0$, $\lim\limits_{n\to+\infty} a_n = \dfrac{\pi}{2}$ *so the sequence* $(a_n z^n)$ *is bounded if and only if* $|z| \leq 1$.
  By definition, we deduce $R = 1$.
- *If* $\alpha = 0$, $a_n = \dfrac{\pi}{4}$ *for all* n *so* $R = 1$.
- *If* $\alpha > 0$, $a_n \underset{n\to+\infty}{\sim} \dfrac{1}{n^\alpha}$ *so the sequence* $(a_n z^n)$ *is bounded if and only if the sequence* $\left(\dfrac{z^n}{n^\alpha}\right)$ *is, i.e., if and only if* $|z| \leq 1$.
  Again, we deduce $R = 1$.

e) *Let* $z \in \mathbb{C}^*$*, we have*

$$\frac{\left|a_{n+1}z^{n+1}\right|}{|a_n z^n|} = \left|\frac{\frac{(2n+2)!}{(n+1)!^2} z^{n+1}}{\frac{(2n)!}{n!^2} z^n}\right| = \frac{(2n+2)(2n+1)}{(n+1)^2}|z| = \frac{2(2n+1)}{n+1}|z| \underset{n\to+\infty}{\longrightarrow} 4|z|$$

, so by applying d'Alembert's rule for series with positive real terms (I won't detail, see ***b)****) we find* $R = \frac{1}{4}\cdot$

f) *We are looking for an equivalent for* a_n*:*

$$\begin{aligned} a_n &= \mathrm{e}^{\frac{1}{n+1}\ln(n+1)} - \mathrm{e}^{\frac{1}{n}\ln(n)} \\ &= \mathrm{e}^{\frac{1}{n}\ln(n)}\left(\mathrm{e}^{\frac{1}{n+1}\ln(n+1) - \frac{1}{n}\ln(n)} - 1\right), \end{aligned}$$

so since $\dfrac{1}{n}\ln(n) \underset{n\to+\infty}{\longrightarrow} 0$ *and* $e^X - 1 \underset{X\to 0}{\sim} X$*:*

$$a_n \underset{n\to+\infty}{\sim} \frac{1}{n+1}\ln(n+1) - \frac{1}{n}\ln(n)$$

and

$$\frac{1}{n+1}\ln(n+1)-\frac{1}{n}\ln(n)=\left(\frac{1}{n+1}-\frac{1}{n}\right)\ln(n)+\underbrace{\frac{1}{n+1}\ln\left(1+\frac{1}{n}\right)}_{\sim\frac{1}{n^2}}\underset{n\to+\infty}{\sim}\frac{\ln(n)}{n^2}.$$

So

$a_n \underset{n\to+\infty}{\sim} \dfrac{\ln(n)}{n^2}$; *it follows that the sequence* $(a_n z^n)$ *is bounded if and only if* $|z|\leq 1$, *so* $R=1$.

g) *The sequence* (a_n) *is periodic and takes only 7 values:* 0, $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, *etc.*
Thus, the sequence $(a_n z^n)$ *is bounded if and only if* $|z|\leq 1$, *and* $R=1$.

h) *The sum of the divisors of* n *is between* 1 *and* $1+2+\cdots+n=\dfrac{n(n+1)}{2}$, *so the radius of convergence of the series* $\sum a_n z^n$ *is between the radii of convergence of the two series* $\sum z^n$ *and* $\sum \dfrac{n(n+1)}{2} z^n$, *both of which are equal to* 1. *Therefore,* $R=1$.

Exercise 3.8.2 *Find the radius of convergence of* $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} z^n$.

Correction 3.8.2 $\displaystyle\lim_{n\to+\infty}\sqrt[n]{|a_n|}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n=e$. *Therefore, the radius of convergence is* $1/e$.

Exercise 3.8.3 *Prove that, if a power series* $\sum a_n x^n$ *converges for* $x=b\neq 0$, *then it converges absolutely for all* x *such that* $|x|<|b|$.

Correction 3.8.3 *Since* $\sum a_n b^n$ *converges,* $\displaystyle\lim_{n\to+\infty}|a_n b^n|=0$. *Since a convergent sequence is bounded, there exists an* M *such that* $|a_n b^n|\leq M$ *for all* n. *Let* $|x/b|=r<1$. *Then* $|a_n x^n|=|a_n b^n|\cdot|x^n/b^n|\leq M r^n$. *Therefore, by comparison with the convergent geometric series* $\Sigma M r^n, \Sigma\,|a_n x^n|$ *is convergent.*

Exercise 3.8.4 *Prove that, if the radius of convergence of* $\Sigma a_n z^n$ *is* R, *then the radius of convergence of* $\Sigma a_n z^{2n}$ *is* $\sqrt{R}$.

Correction 3.8.4 *We denote* R' *as the radius of convergence of* $\sum a_n z^{2n}$.

1. *For* $|z|<\sqrt{R}$, $|z^2|<R$, *and thus* $\sum a_n(z^2)^n=\sum a_n z^{2n}$ *is absolutely convergent. Therefore,* $R'\geq|z|$, *and since this is true for all* z *such that* $|z|<\sqrt{R}$, *we conclude that* $R'\geq\sqrt{R}$.
2. *For* $|z|>\sqrt{R}$, $|z^2|>R$, *and thus* $\sum a_n(z^2)^n=\sum a_n z^{2n}$ *is grossly divergent. Therefore,* $R'\leq|z|$ *and we conclude that* $R'\leq\sqrt{R}$.

Finally: $R'=\sqrt{R}$.

Exercise 3.8.5 *If* $\Sigma a_n z^n$ *has a radius of convergence* r_1 *and if* $\Sigma b_n z^n$ *has a radius of convergence* $r_2>r_1$, *what is the radius of convergence of the sum* $\sum(a_n+b_n)\,z^n$ *?*

Correction 3.8.5 *For* $|z|<r_1$, *both* $\Sigma a_n z^n$ *and* $\Sigma b_n z^n$ *are convergent, and, therefore, so is* $\Sigma\,(a_n+b_n)\,x^n$. *Now, take* z *so that* $r_1<|z|<r_2$. *Then* $\Sigma a_n z^n$ *diverges and* $\Sigma b_n z^n$ *converges. Hence,* $\Sigma\,(a_n+b_n)\,x^n$ *diverges Thus, the radius of convergence of* $\Sigma\,(a_n+b_n)\,z^n$ *is* r_1.

Exercise 3.8.6 *Since the radius of convergence of* $\sum a_n z^n$ *is* $R = 2$*, determine the radius of convergence.*

$$\textbf{\textit{a)}} \sum a_n z^{2n} \qquad \textbf{\textit{b)}} \sum n^4 a_n z^n \qquad \textbf{\textit{c)}} \sum \frac{a_n}{n!} z^n.$$

Correction 3.8.6

a) *According to exercise 6.3.2, the radius of convergence of the series* $\sum a_n z^{2n}$ *is* $\sqrt{2}$.

b) *We showed in class that the power series* $\sum n a_n z^n$ *has the same radius of convergence as* $\sum a_n z^n$*. The series* $\sum n^2 a_n z^n$ *has the same radius of convergence as* $\sum n a_n z^n$*. Continuing in this way, the series* $\sum n^4 a_n z^n$ *has the same radius as* $\sum a_n z^n$*. Therefore, its radius of convergence is* 2.

c) *Given that the radius of convergence of* $\sum a_n z^n$ *is* $R = 2$*, this series converges for all* z *such that* $|z| < R = 2$*. In particular, for* $z = 1$*, the series* $\sum a_n$ *converges. This implies that the general term* a_n *tends to* 0 *as* $n \to +\infty$*. Thus, we have*

$$\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Longrightarrow |a_n| < 1.$$

Therefore,

$$\left|\frac{a_n}{n!} z^n\right| \leq \frac{|z|^n}{n!}, \quad \forall n \in \mathbb{N} : n \geq n_0.$$

It is known, by the ratio test, that the series $\sum \frac{|z|^n}{n!} z^n$ *converges for all* $z \in \mathbb{C}$*. It follows that* $\sum \frac{a_n}{n!} z^n$ *converges absolutely on* $\mathbb{C}$*. Thus, its radius is equal to* $+\infty$.

Exercise 3.8.7 *Find a power series representation for*

$$\frac{1}{(1-x)^2}$$

.

Correction 3.8.7

1. **Method 1**. *For* $|x| < 1$, $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$*. Differentiate this series term by term. Then, for* $|x| < 1$, $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$.

2. **Method 2**. $\dfrac{1}{1-x} \cdot \dfrac{1}{1-x} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{m=0}^{\infty} x^m\right) = \sum_{k=0}^{\infty}\left(\sum_{n+m=k} 1\right) x^k = \sum_{k=0}^{\infty} (k+1) x^k$.

Exercise 3.8.8 *Show that* $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ *for all* x.

Correction 3.8.8 *I Let* $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$*.* $f(x)$ *is defined for all* x*. Differentiate term by term:* $f'(x) = \sum_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = f(x)$*. Moreover,* $f(0) = 1$*. Hence,* $f(x) = e^x$

Exercise 3.8.9 *Give the power series expansion for the following functions:*
1)$f_1(x) = e^{-x^2/2}$, *2)*$f_2(x) = \cosh x$, *3)*$f_3(x) = \sinh x$, *4)*$f_4(x) = \int_0^x e^{-t^2/2}\,\mathrm{dt}$.

Correction 3.8.9

1) *I By exercise 6.4.1,* $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$*. Substitute* $\frac{x^2}{2}$ *for* x*. Then* $e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n \cdot n!}$.

2) *We have* $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{x} \frac{1+(-1)^n}{2} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

3) *Since* $D_x(\cosh x) = \sinh x$*, we can differentiate the power series of Problem* $\cosh x$ *to get* $\sinh x = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

4) *We have* $e^{-t^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n! 2^n}$. *Integrate:* $\int_0^x e^{-t^2/2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \frac{x^{2n+1}}{2n+1}$.

Exercise 3.8.10 *Use power series to solve the differential equation* $y' = -xy$ *under the boundary condition that* $y = 1$ *when* $x = 0$.

Correction 3.8.10 *Let* $y = f(x) = \sum_{n=0}^{\infty} a_n x^n$*. Differentiate:* $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$*. So* $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = -xy = -x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-a_n)\, x^{n+1} = \sum_{n=1}^{\infty} (-a_{n-1})\, x^n$. *Comparing coefficients, we get* $a_1 = 0$*, and* $(n+1) a_{n+1} = -a_{n-1}$*. Since* $y = 1$ *when* $x = 0$*, we know that* $a_0 = 1$*. Now,* $a_3 = a_5 = a_7 = \cdots = 0$*. Also,* $2a_2 = -a_0 = -1$, $a_2 = -\frac{1}{2}$*. Then,* $4a_4 = -a_2$, $a_4 = \frac{1}{2\cdot 4}$*. Further,* $6a_6 = -a_4$, $a_6 = -\frac{1}{2\cdot 4\cdot 6}$*. Similarly,* $8a_8 = -a_6$, $a_8 = \frac{1}{2\cdot 4\cdot 6\cdot 8}$*, and, in general,* $a_{2n} = \frac{(-1)''}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)} = \frac{(-1)^n}{2^n \cdot n!}$*. So* $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}$. *Hence,* $f(x) = e^{-x^2/2}$.

Exercise 3.8.11

1. *Show directly that, if* $y'' = -y$*, and* $y' = 1$ *and* $y = 0$ *when* $x = 0$*, then* $y = \sin x$.

2. *Show that* $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Correction 3.8.11

1. *Let* $z = dy/dx$*. Then* $y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} z$*. Hence,* $\frac{dz}{dy} z = -y$, $\int z dz = -\int y dy$, $\frac{1}{2} z^2 = -\frac{1}{2} y^2 + C$, $z^2 = -y^2 + K$*. Since* $z = 1$ *and* $y = 0$ *when* $x = 0$, $K = 1$*. Thus,* $z^2 = 1 - y^2$, $\frac{dy}{dx} = \pm\sqrt{1-y^2}$, $\int \frac{dy}{\sqrt{1-y^2}} = \pm \int dx, \sin^{-1} y = \pm x + C_1$*. Since* $y = 0$ *when* $x = 0$, $C_1 = 0$*. So* $\sin^{-1} y = \pm x, y = \sin(\pm x) = \pm \sin x$*. Then* $y = \sin x$*. (If* $y = -\sin x$*, then* $y' = -\cos x$*, and* $y' = -1$ *when* $x = 0$*.)*

2. I Let $y = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$. *When* $x = 0$, $y = 0$. *By differentiation,* $y' = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$. *Hence,* $y' = 1$ *when* $x = 0$. *Further,*

$$\begin{aligned} y'' &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\ &= -\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = -y. \end{aligned}$$

Hence, $y = \sin x$.

Exercise 3.8.12 *Find the Taylor series for* $\sin x$ *about* $\pi/4$.

Correction 3.8.12 *Let* $f(x) = \sin x$. *Then* $f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2, f'(\pi/4) = \cos(\pi/4) = \sqrt{2}/2, f''(\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2$. $f'''(\pi/4) = -\cos(\pi/4) = -\sqrt{2}/2$, *and, thereafter, this cycle of four values keeps repeating. Thus, the Taylor series for* $\sin x$ *about* $\frac{\pi}{4}$ *is*

$$\frac{\sqrt{2}}{2}\left[\begin{array}{l} 1 + \frac{x-\pi/4}{1!} - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} \\ \quad + \frac{(x-\pi/4)^4}{4!} + \frac{(x-\pi/4)^5}{5!} - \frac{(x-\pi/4)^6}{6!} - \frac{(x-\pi/4)^7}{7!} + \cdots \end{array}\right]$$

Exercise 3.8.13 *Calculate the Taylor series for* $1/x$ *about 1.*

Correction 3.8.13 *I Let* $f(x) = \frac{1}{x}$. *Then,* $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{2\cdot 3}{x^4}$, $f^{(4)}(x) = \frac{2\cdot 3\cdot 4}{x^5}$, *and, in general,* $f_x^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$. *So* $f^{(n)}(1) = (-1)^n n!$ *Thus, the Taylor series is* $\sum_{n=0}^{x} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}(-1)^n(x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$.

Exercise 3.8.14 *Find the Taylor series for* $\cos x$ *about* $\pi/2$.

Correction 3.8.14 *We have:*

$$\cos x = \sin(\pi/2 - x) = -\sin(x - \pi/2).$$

The Taylor series for $\sin x$ *about* 0 *is:*

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Substitute $x - \pi/2$ *for* x *and multiply by* -1*:*

$$\cos x = -\sum_{n=0}^{\infty}(-1)^n \frac{(x-\pi/2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}.$$

Final Answer

$$\boxed{\cos x = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}}$$

Chapter 4

Fourier Series

In this chapter, we will study Fourier series, which are series of functions of a particular type. You cannot touch a subject of physics, mechanics, electronics, ... without using these series (or their generalization, the Fourier transforms). They are used to study periodic functions: the idea is to express any 2π periodic function as a linear combination (generally, an infinite sum) of simple 2π periodic functions of the form $\cos nx$ or $\sin nx$ with $n \in \mathbb{N}$.

4.1 Trigonometric Series

4.1.1 Periodic Functions

Definition 4.1.1 *A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to have a period T or to be periodic with period T if, for all x, we have $f(x+T) = f(x)$, where T is a positive constant. The smallest value of $T > 0$ is called the least period or simply the period of $f(x)$.*

Example 4.1.2

1. *The functions $x \longrightarrow \sin x$ and $x \longrightarrow \cos x$ are 2π-periodic, and the function $x \longrightarrow e^{(2i\pi/T)x}$ is T-periodic.*
2. *The function $x \longrightarrow \tan x$ is periodic with period π.*
3. *The function $x \mapsto x - \lfloor x \rfloor$ is 1-periodic. It is a well-known function called the* fractional part function *and is usually denoted by $\{x\}$.*

4.1.2 Piecewise Continuous Functions

Definition 4.1.3 *We say that a function f is piecewise continuous on $[a, b]$ when there exist a finite number of points $c_1 = a < c_2 < c_3 < ... < c_{k-1} < c_k = b$ such that f is continuous on each open interval $]c_j, c_{j+1}[$ and admits a finite left and right limit at each c_j. These right and left limits, which we will denote $f\left(c_j^-\right)$ and $f\left(c_j^+\right)$ can be different from the value $f(c_j)$.*

Example 4.1.4 *The floor function $f : x \longrightarrow [x]$ is piecewise continuous on $[0, 4]$ (we can take $c_1 = 0 < c_2 = 1 < c_3 = 2 < c_4 = 3 < c_5 = 4$, and f is continuous on each open interval $]c_j, c_{j+1}[$. Furthermore, the left and right limits of f exist at each c_j, for all $j = 1, ..., 5$).*

4.1.3 Piecewise Differentiable Functions

Definition 4.1.5 *We say that a function f is piecewise differentiable on $[a, b]$ when there exist a finite number of points $c_1 = a < c_2 < c_3 < ... < c_{k-1} < c_k = b$ such that f is differentiable on each open interval $]c_j, c_{j+1}[$ and admits a left and right derivative at each c_j.*

Example 4.1.6 *The functions* $f_1 : x \longrightarrow \sqrt{|x|}$, $f_2 : x \longrightarrow |\sin x|$ *and* $f_3 : x \longrightarrow \dfrac{\sin x}{|x|}$ *are differentiable on* $[-\pi, 0\,[\cup]\,0, \pi]$. *They are not differentiable at 0. The functions* f_2, f_3 *are piecewise differentiable on* $[-\pi, \pi]$ *(it suffices to take* $c_1 = -\pi < c_2 = 0 < c_3 = \pi$*), but the function* f_1 *is not piecewise differentiable on* $[-\pi, \pi]$.

Definition 4.1.7 *The function* f *is said to be piecewise continuous (or piecewise differentiable, respectively) on the interval* I *if and only if it is continuous (or differentiable) on each segment of* I.

4.1.4 2π-periodic Functions on $\mathbb{R}$

A function f defined on $\mathbb{R}$ with period 2π is completely determined by its restriction to a semi-open interval of length 2π. We will often choose $(-\pi, \pi]$ or $[0, 2\pi)$.

To show that a 2π-periodic function f is continuous on $\mathbb{R}$, it suffices to verify:

1. f is continuous on the open interval $(-\pi, \pi)$;
2. f is continuous at π from the left, and the periodic boundary condition holds, i.e.,

$$f(\pi) = \lim_{x \to \pi^-} f(x) = \lim_{x \to -\pi^+} f(x).$$

The equality $\lim_{x \to \pi^-} f(x) = \lim_{x \to -\pi^+} f(x)$ follows from periodicity: as $x \to -\pi^+$, $x + 2\pi \to \pi^-$, and $f(x) = f(x + 2\pi)$. This ensures that f is also right-continuous at $-\pi$ when extended periodically, making f continuous at all points of the form $\pi + 2k\pi$ ($k \in \mathbb{Z}$).

Example 4.1.8 *The* 2π *periodic function defined on* $]-\pi, \pi]$ *by*

$$f(x) = x^2 - \pi^2$$

is continuous on $\mathbb{R}$. *Indeed, this function is clearly continuous on* $]-\pi, \pi[$, *as a polynomial. Moreover, we have*

$$f(\pi) = \lim_{x \to \pi^-} f(x) = \lim_{x \to -\pi^+} f(x) = 0.$$

This shows that f *is continuous at* π *and therefore everywhere on* $\mathbb{R}$.

Example 4.1.9 *The* 2π *periodic function defined on* $]-\pi, \pi]$ *by*

$$f(x) = x$$

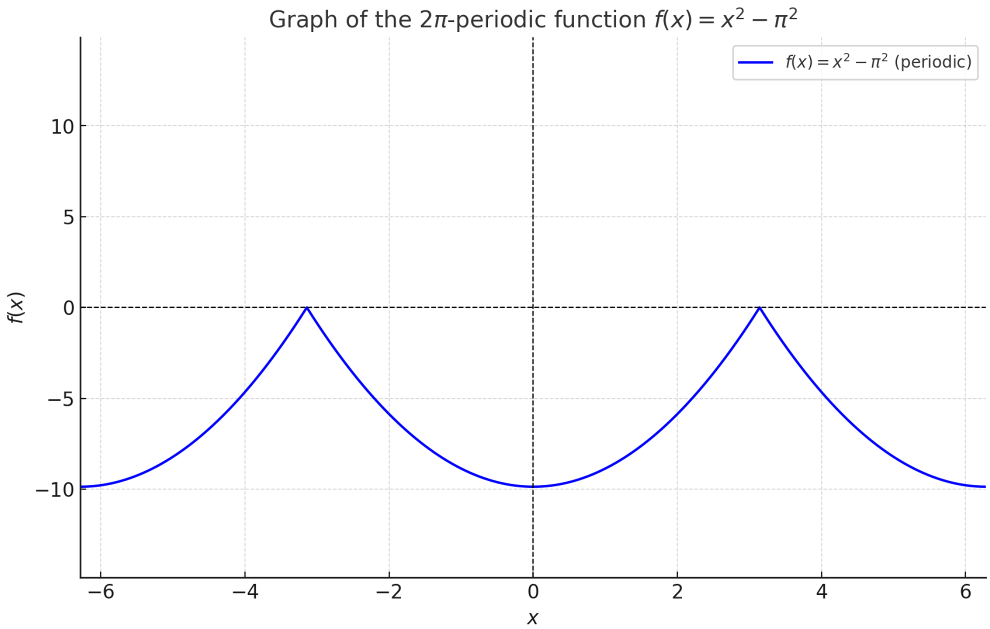
is not continuous on $]-\pi, \pi]$, *but only piecewise continuous. Indeed, this function is a polynomial, so it is continuous on* $]-\pi, \pi[$, *as a polynomial. On the other hand,*

$$f(\pi) = \lim_{x \to \pi^-} f(x),$$

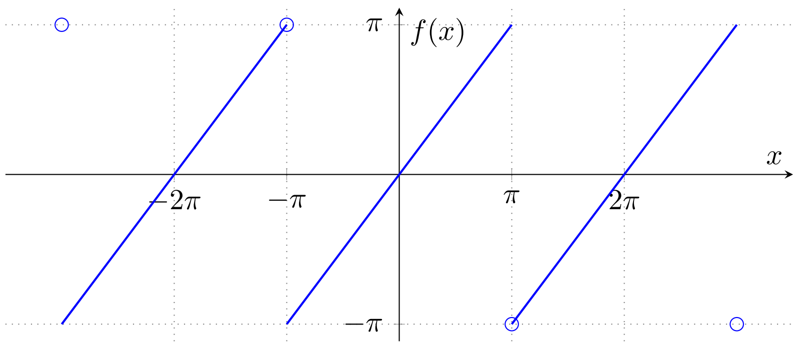
and,

$$\lim_{x \to -\pi^+} f(x) = -\pi \neq f(\pi).$$

Therefore, f *is not continuous at* $x = \pi$. *But since it admits a left limit and a right limit there, it is piecewise continuous on* $]-\pi, \pi]$.



Graph of the function $f(x) = x$ defined modulo 2π



4.1.5 2π-periodic Functions of Class C^1 on $\mathbb{R}$

To check that a 2π-periodic function is of class C^1 on $]-\pi,\pi]$, i.e., on $\mathbb{R}$ as a whole, we must verify that:

1. f is continuous on $]-\pi,\pi[$,
2. f is continuously differentiable on $]-\pi,\pi[$,
3. $f(\pi) = \lim_{x\to\pi^-} f(x) = \lim_{x\to-\pi^+} f(x)$,
4. $\lim_{x\to\pi^-} f'(x) = \lim_{x\to-\pi^+} f'(x)$.

The last equality means that f admits a left derivative equal to the right derivative at $x = \pi$, and that it is therefore differentiable on $\mathbb{R}$.

Example 4.1.10 *The 2π periodic function defined on $]-\pi,\pi]$ by*

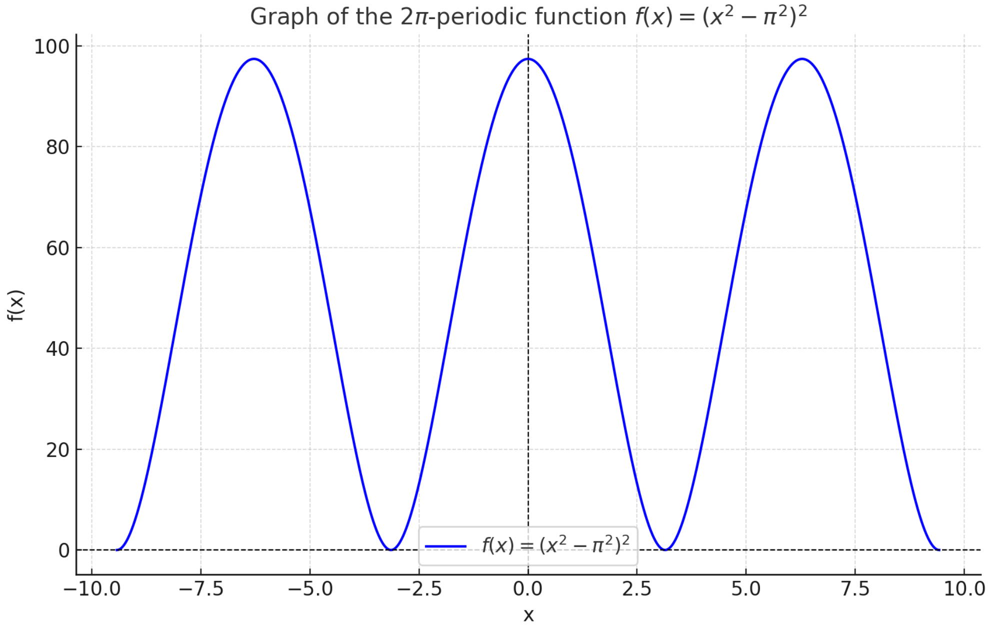
$$f(x) = (x^2 - \pi^2)^2$$

is of class C^1 on $\mathbb{R}$. Indeed, the first three conditions are satisfied, and we have

$$\lim_{x\to-\pi^+} \frac{f(x) - f(-\pi)}{x - (-\pi)} = \lim_{x\to-\pi^+} \frac{(x^2-\pi^2)^2}{x+\pi} = 0,$$

and,

$$\lim_{x\to\pi^-}\frac{f(x)-f(\pi)}{x-\pi}=\lim_{x\to\pi^-}\frac{(x^2-\pi^2)^2}{x-\pi}=0.$$



Example 4.1.11 *The* 2π *periodic function defined on* $]-\pi,\pi]$ *by*

$$f(x)=x^2-\pi^2$$

is not differentiable on $\mathbb{R}$*, so it is not* C^1*. Indeed, it is easy to verify the first three conditions, but the last one is not satisfied because*

$$\lim_{x\to-\pi^+}\frac{f(x)-f(-\pi)}{x-(-\pi)}=-2\pi,$$

and,

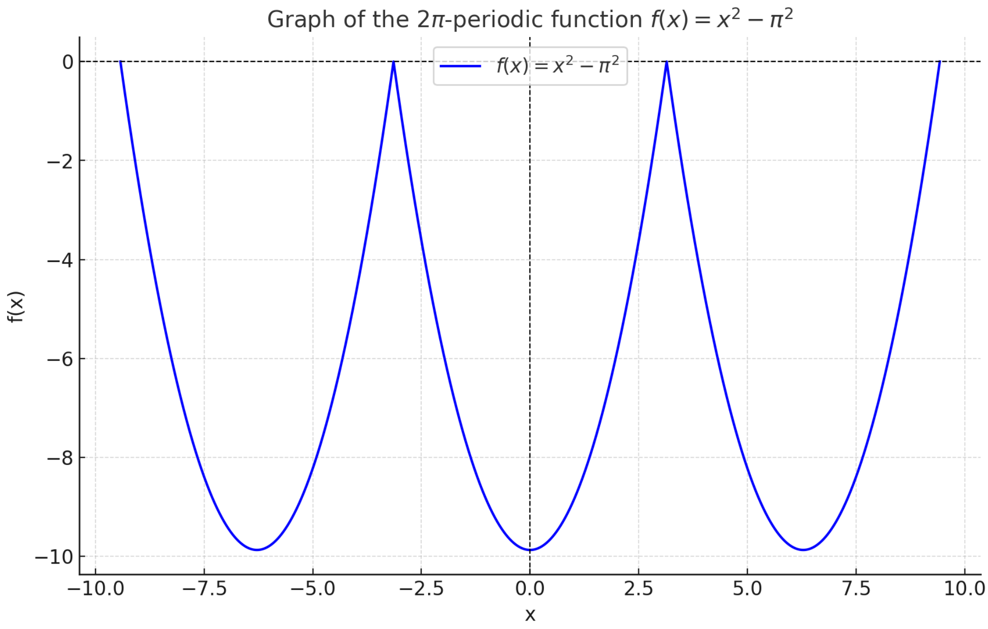
$$\lim_{x\to\pi^-}\frac{f(x)-f(\pi)}{x-\pi}=2\pi.$$

Hence, f *is not* C^1*; it is only piecewise* C^1*.*

We recall the following elementary property that is constantly used.

Proposition 4.1.12 *Let* f *be a* T*-periodic function. If* f *is piecewise continuous on* $[0,T]$ *then for all* $x_0\in\mathbb{R}$*,* f *is piecewise continuous on* $[x_0,x_0+T]$ *and we have*

$$\int_{x_0}^{x_0+T}f(t)dt=\int_0^T f(t)dt.$$



Proof: The Chasles relation allows us to write:

$$\int_{x_0}^{x_0+T} f(t)dt = \int_{x_0}^{0} f(t)dt + \int_{0}^{T} f(t)dt + \int_{T}^{x_0+T} f(t)dt$$

In the integral $\int_{T}^{x_0+T} f(t)dt$ we make the change of variables $y = t - T$.
This gives us

$$\int_{T}^{x_0+T} f(t)dt = \int_{0}^{x_0} f(y+T)dy = \int_{0}^{x_0} f(y)dy.$$

Therefore

$$\int_{x_0}^{x_0+T} f(t)dt = \int_{x_0}^{0} f(t)dt + \int_{0}^{T} f(t)dt + \int_{0}^{x_0} f(t)dt = \int_{0}^{T} f(t)dt.$$

□

Remark 4.1.13 *For any function f assumed to be 2π-periodic and piecewise continuous, Proposition 4.1.12 gives in particular the relation*

$$\int_{0}^{2\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.$$

4.1.6 Trigonometric Series

Definition 4.1.14 *We call a trigonometric series a series of functions: $\sum u_n(x)$ whose general term $u_n(x)$ is of the form*

$$u_0(x) = \frac{a_0}{2} \text{ and } \forall n \in \mathbb{N}^* \; u_n(x) = a_n \cos(nx) + b_n \sin(nx)$$

where $a_n, b_n \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ are called the coefficients of the series, and $x \in \mathbb{R}$.
We agree to write a trigonometric series in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)), \tag{4.1}$$

where we set $b_0 = 0$.

Definition 4.1.15 *A function $f : \mathbb{R} \longrightarrow \mathbb{C}$, 2π-periodic, is expandable in a trigonometric series if it is the simple limit (or sum) of a trigonometric series.*

Example 4.1.16

1. *The functions sin and cos are expandable in trigonometric series (determine the corresponding coefficients a_n and b_n).*
2. *The series with general term $\dfrac{\cos(nx)}{n^2}$ is trigonometric.*

Example 4.1.17 *The series $\sum_{n\geq 0} \dfrac{e^{inx}}{n!}$ is a trigonometric series. Indeed,*

$$\sum_{n\geq 0} \frac{e^{inx}}{n!} = \sum_{n\geq 0}\left(\frac{1}{n!}\cos nx + \frac{i}{n!}\sin nx\right),$$

with $a_n = \dfrac{1}{n!}$ and $b_n = \dfrac{i}{n!}$.

4.1.7 Convergence of a Trigonometric Series

The study of the convergence of trigonometric series is not easy in general. We will be content with the following results.

Proposition 4.1.18

If the infinite series $\left(\sum a_n\right)$ and $\left(\sum b_n\right)$ are absolutely convergent then the trigonometric series (4.1) is normally convergent on $\mathbb{R}$; therefore absolutely and uniformly on $\mathbb{R}$ and its sum S is $2\pi -$ periodic.

Proof: The series $\sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ is normally convergent on $\mathbb{R}$. Indeed, this follows from the inequality:

$$|a_n \cos(nx) + b_n \sin(nx)| \leq |a_n| + |b_n| \quad \forall x \in \mathbb{R}.$$

And from the fact that $\left(\sum a_n\right)$ and $\left(\sum b_n\right)$ are absolutely convergent, therefore the series $\left(\sum |a_n| + |b_n|\right)$ is convergent.
We now denote S the sum of the series (4.1). We verify that S is 2π-periodic i.e.,

$$S(x + 2\pi) = S(x) \quad \forall x \in \mathbb{R}.$$

We have,

$$S(x + 2\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n(x + 2\pi) + b_n \sin n(x + 2\pi)).$$

And since,

$$\cos(nx + 2n\pi) = \cos nx \text{ and } \sin(nx + 2n\pi) = \sin nx,$$

therefore,

$$S(x + 2\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = S(x) \quad \forall x \in \mathbb{R}..$$

□

Remark 4.1.19 *We note that if* $\sum_{n\geq 0}(|a_n|+|b_n|)$ *is convergent then the series (4.1) is uniformly convergent on* $\mathbb{R}$.

Proposition 4.1.20 *If the numerical sequences* (a_n) *and* (b_n) *are decreasing and tend to* 0, *then the trigonometric series (4.1) converges for all* $x \neq 2k\pi$ *where* $k \in \mathbb{Z}$.

Proof: This follows from Abel's (or Dirichlet's) test for convergence. It suffices to show that the partial sums

$$S(m,n) = \sum_{p=m}^{n} \sin px \quad \text{and} \quad C(m,n) = \sum_{p=m}^{n} \cos px$$

are bounded independently of m and n for $x \neq 2k\pi$.

Let us first compute closed forms for $S_n = \sum_{p=0}^{n} \sin pt$ and $C_n = \sum_{p=0}^{n} \cos pt$ for $t \neq 2k\pi$. Consider the geometric sum:

$$C_n + iS_n = \sum_{p=0}^{n} e^{ipt} = \frac{1-e^{i(n+1)t}}{1-e^{it}}.$$

Write $1-e^{i\theta} = -2ie^{i\theta/2}\sin(\theta/2)$. Then

$$C_n + iS_n = \frac{-2ie^{i(n+1)t/2}\sin\frac{(n+1)t}{2}}{-2ie^{it/2}\sin\frac{t}{2}} = e^{int/2}\frac{\sin\frac{(n+1)t}{2}}{\sin\frac{t}{2}}.$$

Hence, taking real and imaginary parts,

$$C_n = \frac{\sin\frac{(n+1)t}{2}\cos\frac{nt}{2}}{\sin\frac{t}{2}}, \qquad S_n = \frac{\sin\frac{(n+1)t}{2}\sin\frac{nt}{2}}{\sin\frac{t}{2}}.$$

Now, for $x \neq 2k\pi$, let $t = x$. Then

$$|S(m,n)| = \left|\sum_{p=m}^{n} \sin px\right| = |S_n - S_{m-1}| \leq |S_n| + |S_{m-1}| \leq \frac{1}{|\sin(x/2)|} + \frac{1}{|\sin(x/2)|} = \frac{2}{|\sin(x/2)|},$$

because $|\sin\dfrac{(n+1)x}{2}\sin\dfrac{nx}{2}| \leq 1$. Similarly, $|C(m,n)| \leq \dfrac{2}{|\sin(x/2)|}$.

Since a_n and b_n are decreasing and tend to 0, and since the partial sums $\sum_{p=m}^{n} \cos px$ and $\sum_{p=m}^{n} \sin px$ are bounded uniformly in m, n, Abel's test implies convergence of

$$\sum_{n=0}^{\infty}\big(a_n \cos nx + b_n \sin nx\big)$$

for all $x \neq 2k\pi$, $k \in \mathbb{Z}$. □

4.1.8 Complex Representation of a Trigonometric Series

From Euler's formulas:

$$\cos(nx) = \frac{\mathrm{e}^{inx} + \mathrm{e}^{-inx}}{2} \text{ and } \sin(nx) = \frac{\mathrm{e}^{inx} - \mathrm{e}^{-inx}}{2i}$$

the series (4.1) becomes:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \frac{\mathrm{e}^{inx} + \mathrm{e}^{-inx}}{2} + b_n \frac{\mathrm{e}^{inx} - \mathrm{e}^{-inx}}{2i} \right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\mathrm{e}^{inx} \frac{a_n - ib_n}{2} + \mathrm{e}^{-inx} \frac{a_n + ib_n}{2} \right]$$

By setting:
$c_n = \dfrac{a_n - ib_n}{2}$; $\quad c_{-n} = \bar{c}_n = \dfrac{a_n + ib_n}{2}$ and $c_0 = \dfrac{a_0}{2}$, the series becomes:

$$\begin{aligned} c_0 + \sum_{n=1}^{\infty} \left(c_n \mathrm{e}^{inx} + c_{-n} \mathrm{e}^{-inx} \right) &= c_0 + \sum_{n=1}^{\infty} c_n \mathrm{e}^{inx} + \sum_{n=1}^{\infty} c_{-n} \mathrm{e}^{-inx} \\ &= c_0 + \sum_{n=1}^{\infty} c_n \mathrm{e}^{inx} + \sum_{n=-\infty}^{-1} c_n \mathrm{e}^{inx} = \sum_{n \in \mathbb{Z}} c_n \mathrm{e}^{inx}. \end{aligned}$$

This last expression is called the complex form of a trigonometric series.
Calculating the Coefficients of the Trigonometric Series. Real Case
Let's consider the conditions of uniform convergence of the trigonometric series (4.1) and set

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right).$$

Then

$$\begin{aligned} f(x)\cos(nx) &= \frac{a_0}{2}\cos(nx) + \sum_{k=1}^{\infty} \left[a_k \cos(kx)\cos(nx) + b_k \sin(kx)\cos(nx) \right] \\ f(x)\sin(nx) &= \frac{a_0}{2}\sin(nx) + \sum_{k=1}^{\infty} \left[a_k \cos(kx)\sin(nx) + b_k \sin(kx)\sin(nx) \right] \end{aligned}$$

Uniform convergence allows us to have:

$$\begin{aligned} \int_0^{2\pi} f(x)\cos(nx)dx &= \frac{a_0}{2} \int_0^{2\pi} \cos(nx)dx + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kx)\cos(nx)dx \\ &\quad + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kx)\cos(nx)dx \\ \int_0^{2\pi} f(x)\sin(nx)dx &= \frac{a_0}{2} \int_0^{2\pi} \sin(nx)dx + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kx)\sin(nx)dx \\ &\quad + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kx)\sin(nx)dx. \end{aligned}$$

But we have:

$$\int_0^{2\pi} \cos(kx)\cos(nx)dx = \begin{cases} 0 & \text{if} \quad k \neq n \\ \\ \pi & \text{if} \quad k = n \end{cases}$$

$$\int_0^{2\pi} \sin(kx)\sin(nx)dx = \begin{cases} 0 & \text{if} \quad k \neq n \\ \\ \pi & \text{if} \quad k = n \end{cases}$$

$$\int_0^{2\pi} \cos(nx)\sin(kx)dx = 0$$

We then deduce the coefficients by the following expressions:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx.$$

These expressions are valid even for $n = 0$. By proposition 4.1.12, the coefficients can be written:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\cos(nx)dx \quad \forall\alpha \in \mathbb{R}.$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin(nx)dx; \quad \forall\alpha \in \mathbb{R}.$$

Case of 2π-periodic functions;

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Calculating the Coefficients of the Trigonometric Series. Complex Case

We have $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$.

$$\int_0^{2\pi} f(x)e^{-inx}dx = \sum_{k=-\infty}^{\infty} c_k \int_0^{2\pi} e^{ix(k-n)}dx.$$

Now,

$$\int_0^{2\pi} e^{ix(k-n)}dx = \begin{cases} 0 & \text{if} \quad k \neq n \\ \\ 2\pi & \text{if} \quad k = n \end{cases}$$

The coefficients are then given by the relation:

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_\alpha^{\alpha+2\pi} f(x)e^{-inx}; n \in \mathbb{Z}.$$

Fourier Series

So far, we have been interested in studying trigonometric series whose coefficients are known from the start.
In this section, we will be interested in the inverse problem. We are given a function $f : \mathbb{R} \mapsto \mathbb{R}$, 2π-periodic. Under what additional conditions does it admit, in a domain D of $\mathbb{R}$ (to be specified), a development in trigonometric series, and in the case where such a development exists, how to determine D as well as the coefficients a_n and b_n corresponding?

4.1.9 Fourier Series of a Periodic Function

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic function that is locally integrable on $\mathbb{R}$, i.e., f is integrable over any interval $[a, b] \subset \mathbb{R}$.

Definition 4.1.21 *The Fourier series associated with f is the trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where the coefficients a_n and b_n are given by:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx, \quad \forall n \in \mathbb{N}$$
$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx, \quad \forall n \in \mathbb{N}.$$

The numbers a_n and b_n are called the Fourier coefficients of the function f.

Notation 4.1.22 *When the Fourier series of the function f converges at $x \in \mathbb{R}$, we denote its sum by $S_f(x)$, i.e.,*

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Remark 4.1.23 *If f is a 2π-periodic function that is locally integrable on $\mathbb{R}$, f is not necessarily the sum of the Fourier series. That is, we may have $S_f \neq f$, as the following example shows.*

Example 4.1.24 *Let f be a 2π-periodic function such that:*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [0, \pi[\\ \\ -1 & \text{if} \quad x \in]-\pi, 0[\end{cases}$$

To determine the Fourier series of f, we will calculate its coefficients. We have from proposition 4.1.12

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{0} -1 dx + \frac{1}{\pi} \int_0^{\pi} 1 dx = 0.$$

After calculation, we obtain

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0,$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx,$$

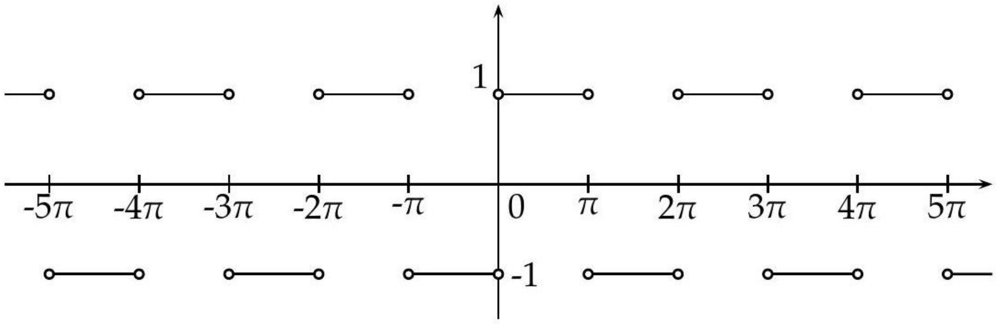


Figure 4.1: Graph of the function $f(x)$.

Thus,

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nt)dt = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \textit{if} \quad n \quad \textit{is even} \\ \\ \frac{4}{n\pi} & \textit{if} \quad n \quad \textit{is odd} \end{cases}$$

Consequently, the associated Fourier series is:

$$S_f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}, \quad \forall x \in \mathbb{R}.$$

For $x = 0$ we have

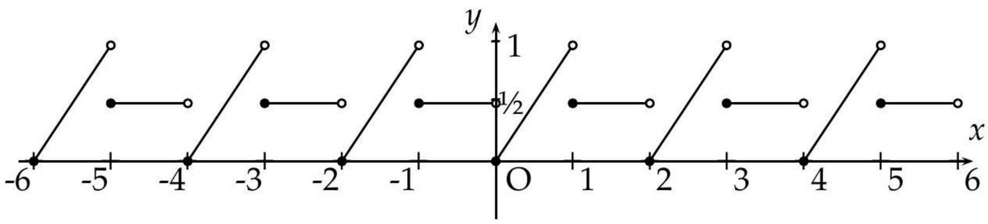


Figure 4.2: Graph of the function $S_f(x)$.

$$S_f(0) = 0 \neq f(0) = 1.$$

This justifies that the sum of the Fourier series of f does not coincide with f.

Thus, we have the following theorem.

Theorem 4.1.25 *If the series $\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$ converges uniformly on $\mathbb{R}$, then it is the Fourier series of its sum.*

Proof: Let's show that the series $\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$ is the Fourier series of its sum S, i.e.,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\cos nx dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin nx dx$$

with $S(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$, $x \in \mathbb{R}$. By hypothesis, the trigonometric series converges uniformly, and since the functions $x \longrightarrow f_n(x) = a_n \cos nx + b_n \sin nx$ are continuous, S is continuous. Furthermore, S is 2π-periodic.
We multiply $S(x)$ by $\cos mx$ and integrate the results term by term over $[-\pi, \pi]$, we then obtain,

$$\int_{-\pi}^{\pi} S(x) \cos mx dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{+\infty} (a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx$$
$$+ b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx).$$

Now,

$$\int_{-\pi}^{\pi} \cos(nx) \cos(mx) dx = \begin{cases} 0 & \text{if} \quad n \neq m \\ \\ \pi & \text{if} \quad n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx) \sin(nx) dx = 0$$

and,

$$\int_{-\pi}^{\pi} \cos(mx) dx = 0.$$

Hence,

$$\int_{-\pi}^{\pi} S(x) \cos mx dx = \pi a_m.$$

Therefore,

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} S(x) \cos mx dx.$$

We multiply $S(x)$ by $\sin mx$ and integrate the results term by term over $[-\pi, \pi]$, we then obtain,

$$\int_{-\pi}^{\pi} S(x) \sin mx dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx dx + \sum_{n=1}^{+\infty} (a_n \int_{-\pi}^{\pi} \cos nx \sin mx dx$$
$$+ b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx).$$

After calculation, we obtain

$$\int_{-\pi}^{\pi} S(x) \sin mx dx = \pi b_m.$$

Therefore,

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} S(x) \sin mx dx.$$

□

4.1.10 Fourier Series of Even and Odd Functions

In what follows, we will frequently need the following well-known result involving the parity of a function.

Lemma 4.1.26 *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally integrable function. Then,* *1. if g is odd, then for all $a > 0$,*

$$\int_{-a}^{a} g(x)dx = 0,$$

2. if g is even, then for all $a > 0$,

$$\int_{-a}^{a} g(x)dx = 2\int_{0}^{a} g(x)dx.$$

As a consequence, we have:

Corollary 4.1.27 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic and locally integrable function. Then,*

1. If f is odd, then the Fourier coefficients of f are:

$$a_n = 0, \quad \forall n \in \mathbb{N}.$$
$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx.$$

2. If f is even, then the Fourier coefficients of f are:

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)dx.$$
$$b_n = 0, \quad \forall n \in \mathbb{N}.$$

Thus, the Fourier series of an even function contains only cosines, and the Fourier series of an odd function contains only sines.

Proof: If f is expandable in a Fourier series:

1. If f is odd:

- $a_n = 0$ since the function $x \mapsto f(x)\cos(nx)$ is odd.
- $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\sin(nx)dx$ since the function $x \mapsto f(x)\sin(nx)$ is even.

2. If f is even:

- $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\cos(nx)dx$

since the function $x \mapsto f(x)\cos(nx)$ is even.

- $b_n = 0$ since the function $x \mapsto f(x)\sin(nx)$ is odd.

□

Example 4.1.28 *Let f be a 2π-periodic function such that:*

$$f(x) = |x| \quad \text{if} \quad x \in [-\pi, \pi]$$

Since f is even, $\Longrightarrow b_n = 0, \forall n \in \mathbb{N}$
We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|dt = \frac{2}{\pi}\int_{0}^{\pi} tdt = \frac{2}{\pi}\left(\frac{t^2}{2}\right)_0^{\pi} = \pi$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos ntdt = \frac{2}{\pi}\int_0^{\pi} t\cos ntdt = \frac{2}{\pi}\left(\left[\frac{t\sin nt}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}dt\right)$$
$$= \frac{-2}{n\pi}\int_0^{\pi}\sin ntdt = \frac{2}{n\pi}\left(\frac{\cos nt}{n}\right)_0^{\pi} = \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-4}{\pi n^2} & \text{if } n \text{ is odd}\end{cases}$$

Thus, the sum of the Fourier series of f is given by:

$$S(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}, \quad x \in \mathbb{R}$$

since the series $\sum_{n\geq 1}|a_n| = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$ *converges.*
Now, the series $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$ *converges uniformly on $\mathbb{R}$, so according to Theorem 4.1.25, we have $f(x) = S(x), \forall x \in \mathbb{R}$.*

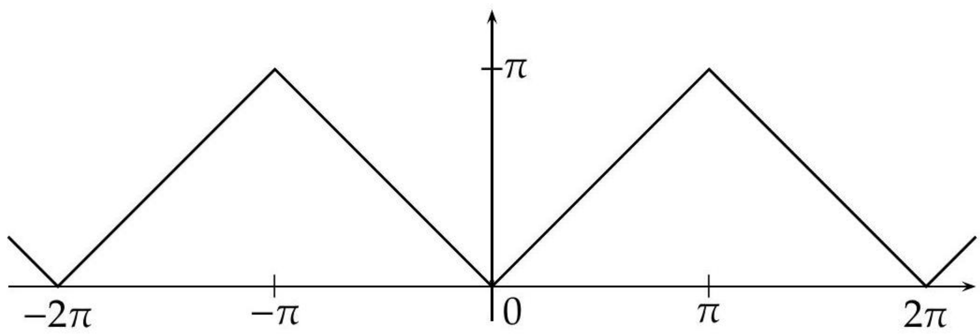


Figure 4.3: Graphs of the functions $f(x)$ and $S(x)$.

Remark 4.1.29

1. *for $x = 0$ we have $f(0) = 0$ and since $f(0) = S(0)$ we have:*

$$S(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Longleftrightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

2. *By writing*

$$S = \sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

We then deduce:

$$\frac{3S}{4} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

or

$$S = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Pointwise Convergence of Fourier Series: Dirichlet's Theorem

Let $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ be a Fourier series of a function f. We ask the following questions:

1. Does the Fourier series associated with f converge?
2. If it converges, can we say that the series converges to f?
3. If this series converges without converging to $f(x)$, $x \in \mathbb{R}$. What is the sum of this series?

The following theorem, called Dirichlet's theorem, answers the above questions.
Before stating the theorem, let us recall the notion of a discontinuity of the first kind and the Riemann-Lebesgue lemma.

Definition 4.1.30 *A function f has a discontinuity of the first kind at a point x_0 if the right and left limits of x_0 exist. (These are not necessarily equal except in the case of continuity.)*

Lemma 4.1.31 (Riemann-Lebesgue Lemma) *Let I be an interval and $f : \mathbb{R} \mapsto \mathbb{C}$ integrable. Then*

$$\lim_{n\to\pm\infty} \int_I f(t) \cos ntdt = 0 \text{ and } \lim_{n\to\pm\infty} \int_I f(t) \sin ntdt = 0.$$

Remark 4.1.32 *This lemma makes it possible to demonstrate the decay to 0 of the Fourier coefficients. It is particularly easy to prove if f is of class C^1 on $[a,b]$: it suffices to perform integration by parts!*
More generally, under the same assumptions,

$$\lim_{s\to\pm\infty} \int_I f(t)e^{-ist}dt = 0.$$

Theorem 4.1.33 (Dirichlet's Theorem) *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a periodic function with period $T = 2\pi$ satisfying the following conditions (called Dirichlet's conditions):*

D1) The discontinuities of f (if they exist) are of the first kind and are finite in number in any finite interval.

D1) f has a right derivative and a left derivative at every point.

Then the Fourier series associated with f is convergent and we have:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$
$$= \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{f(x_0^+)+f(x_0^-)}{2} & \text{if } f \text{ is discontinuous at } x \end{cases}$$

Moreover, the convergence is uniform on any interval where the function f is continuous.

The notations $f(x_0^+)$ and $f(x_0^-)$ represent the right and left limits of f at the point x, respectively.

Proof: Let $S_n(x_0)$ be the partial sum of order n of the Fourier series of f defined by

$$S_n(x_0) = \frac{a_0}{2} + \sum_{p=1}^{n}(a_p \cos(px_0) + b_p \sin(px_0)).$$

Taking into account the expression of the Fourier coefficients of f, we have

$$\begin{aligned} S_n(x_0) = & \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt \\ & + \sum_{p=1}^{n} \frac{1}{\pi}\left(\cos px_0 \int_{-\pi}^{\pi} f(t)\cos pt dt + \sin px_0 \int_{-\pi}^{\pi} f(t)\sin pt dt\right). \end{aligned}$$

Thus,

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{p=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi}(\cos px_0 \cos pt + \sin px_0 \sin pt)f(t)dt.$$

Using trigonometric formulas, we obtain

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{p=1}^{n}\cos p(t-x_0)\right)dt. \tag{4.2}$$

And since,

$$1 + 2\sum_{p=1}^{n}\cos p(t-x_0) = \frac{[\sin[(2n+1)\left(\frac{t-x_0}{2}\right)]}{\sin\left(\frac{t-x_0}{2}\right)}.$$

Therefore, equation (4.2) becomes

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{[\sin[(2n+1)\left(\frac{t-x_0}{2}\right)]}{\sin\left(\frac{t-x_0}{2}\right)}dt. \tag{4.3}$$

Making a change of variable $u = t - x_0$ in (4.3), we obtain

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi-x_0}^{\pi-x_0} f(u+x_0)\frac{[\sin[(2n+1)\left(\frac{u}{2}\right)]}{\sin\left(\frac{u}{2}\right)}du. \tag{4.4}$$

Since the function to be integrated is 2π-periodic because f and sine are, and it is piecewise continuous on $]-\pi,\pi]$ because f is piecewise differentiable and $u \longrightarrow \dfrac{\sin[(2n+1)\left(\frac{u}{2}\right)}{\sin\left(\frac{u}{2}\right)}$ is C^1 on $]-\pi,\pi]$. Therefore, by application of proposition 4.1.12, (4.4) becomes

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u+x_0)\frac{[\sin(2n+1)\left(\frac{u}{2}\right)]}{\sin\left(\frac{u}{2}\right)}du.$$

This can be written as the sum of two integrals as follows:

$$S_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{0} f(u+x_0)\frac{[\sin[(2n+1)\left(\frac{u}{2}\right)]}{\sin\left(\frac{u}{2}\right)}du$$
$$+\frac{1}{2\pi}\int_{0}^{\pi} f(u+x_0)\frac{[\sin[(2n+1)\left(\frac{u}{2}\right)]}{\sin\left(\frac{u}{2}\right)}du.$$

Making a change of variable $u = -u$ in the first integral, we obtain

$$S_n(x_0) = \frac{1}{2\pi}\int_{0}^{\pi} (f(x_0-u)+f(x_0+u))\frac{[\sin[(2n+1)\left(\frac{u}{2}\right)]}{\sin\left(\frac{u}{2}\right)}du.$$

We now set

$$y_0 = \frac{f(x_0^-)+f(x_0^+)}{2},$$

Then

$$S_n(x_0) - y_0 = \frac{1}{2\pi}\int_{0}^{\pi} g(u)\sin[(2n+1)\left(\frac{u}{2}\right)]du,$$

where g is a mapping from $]0,\pi[$ to $\mathbb{R}$ defined by

$$g(u) = \frac{f(x_0-u)-f(x_0^-)+f(x_0+u)-f(x_0^+)}{\sin\frac{u}{2}}.$$

Since by hypothesis f is piecewise differentiable, the following two limits exist:

$$\beta = \lim_{u\to 0}\frac{f(x_0-u)-f(x_0^-)}{u} \quad \text{and} \quad \alpha = \lim_{u\to 0}\frac{f(x_0+u)-f(x_0^+)}{u}.$$

Consequently

$$\lim_{u\to 0} g(u) = 2(\beta+\alpha).$$

Thus, the mapping g can be extended by continuity at 0. Since f is 2π-periodic and piecewise differentiable on an interval of length 2π, the mapping g is piecewise continuous on $[0,\pi]$ and therefore Riemann integrable on $[0,\pi]$. We deduce from the Riemann-Lebesgue lemma that

$$\lim_{n\to+\infty}\frac{1}{2\pi}\int_{0}^{\pi} g(u)\sin[(2n+1)(\frac{u}{2})du = 0.$$

Therefore,

$$\lim_{n\to+\infty}(S_n(x_0)-y_0) = 0.$$

Thus, the sequence $(S_n(x_0))_n$ converges and has limit y_0. If, in particular, f is continuous at x_0, then

$$f(x_0^-) = f(x_0^+) = f(x_0).$$

Consequently,

$$y_0 = \frac{f(x_0^-)+f(x_0^+)}{2} = f(x_0),$$

and $(S_n(x_0))_n$ converges and has limit $f(x_0)$. Hence, Dirichlet's theorem is proved. □

Remark 4.1.34 *There is another theorem equivalent to Theorem 4.1.33 due to Jordan.*

Theorem 4.1.35 (Jordan's Theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function satisfying:*

J1) f is bounded: $\exists M > 0$ such that $|f(x)| \le M$ for all x.

J2) On any interval of length 2π, f has bounded variation (equivalently, f can be written as the difference of two bounded increasing functions on such intervals).

Then the Fourier series of f converges pointwise to

$$S(x) = \frac{f(x^+) + f(x^-)}{2}$$

at every $x \in \mathbb{R}$, where $f(x^+)$ and $f(x^-)$ denote the right-hand and left-hand limits of f at x. In particular, if f is continuous at x, then $S(x) = f(x)$.

Moreover, if f is continuous on a closed interval $[a, b]$, then the convergence of the Fourier series is uniform on $[a, b]$.

Proof: We outline the key steps; a full detailed proof requires several lemmas about functions of bounded variation and their Fourier series.

Step 1: Reduction to a single point. By periodicity, it suffices to study convergence on $[-\pi, \pi]$. Fix $x_0 \in (-\pi, \pi)$. The N-th partial sum of the Fourier series is

$$S_N(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) D_N(x_0 - x)\, dx,$$

where $D_N(t) = \frac{\sin((N+1/2)t)}{2\sin(t/2)}$ is the Dirichlet kernel. By periodicity and a change of variables $t = x_0 - x$, we obtain

$$S_N(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0 - t) D_N(t)\, dt.$$

Since $\frac{1}{\pi} \int_{-\pi}^{\pi} D_N(t)\, dt = 1$, we can write

$$S_N(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{1}{\pi} \int_{0}^{\pi} \left[f(x_0 + t) + f(x_0 - t) - f(x_0^+) - f(x_0^-) \right] \frac{\sin((N + 1/2)t)}{2\sin(t/2)}\, dt.$$

Step 2: Use of condition (J2) – bounded variation. The hypothesis (J2) means f has bounded variation on $[x_0 - \pi, x_0 + \pi]$. A key property: if g is of bounded variation on $[a, b]$, then

$$\lim_{N \to \infty} \int_a^b g(t) \sin(Nt)\, dt = 0.$$

Indeed, by integration by parts (Riemann-Stieltjes sense) or by approximating g by step functions, one shows the Fourier coefficients of a function of bounded variation are $O(1/n)$.

Write

$$\varphi(t) = \frac{f(x_0 + t) - f(x_0^+) + f(x_0 - t) - f(x_0^-)}{2\sin(t/2)}.$$

The function $\varphi(t)$ is of bounded variation on $[0, \pi]$ because:

- $f(x_0 + t)$ and $f(x_0 - t)$ are of bounded variation in t,
- $2\sin(t/2)$ is smooth and bounded away from 0 on $[\delta, \pi]$, and near $t = 0$,

$$\frac{f(x_0 + t) - f(x_0^+)}{2\sin(t/2)} = \frac{f(x_0 + t) - f(x_0^+)}{t} \cdot \frac{t}{2\sin(t/2)}$$

  and the first factor is of bounded variation because f is of bounded variation (its derivative in the distribution sense is a finite measure).

Step 3: Application of the Riemann-Lebesgue lemma for BV functions. Since φ is of bounded variation on $[0, \pi]$, by the Riemann-Lebesgue lemma for BV functions,

$$\lim_{N\to\infty} \int_0^{\pi} \varphi(t) \sin((N+1/2)t)\, dt = 0.$$

But

$$S_N(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} = \frac{1}{\pi} \int_0^{\pi} \varphi(t) \sin((N+1/2)t)\, dt.$$

Hence

$$\lim_{N\to\infty} S_N(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Step 4: Uniform convergence on intervals of continuity. If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous there. The Dirichlet kernel convolution formula can be analyzed more carefully; one uses the fact that the Fejér means converge uniformly for continuous periodic functions (Fejér's theorem), and then uses the uniform boundedness of partial sums for functions of bounded variation (using the Dirichlet-Jordan test). A precise argument employs the fact that the Fourier series is uniformly Cesàro summable to f on $[a, b]$, and for functions of bounded variation, the Fourier series itself (not just Cesàro means) converges uniformly on any closed interval of continuity. □

Remark 4.1.36 *Condition (J2) as originally stated in some formulations ("piecewise monotone") is a special case of bounded variation. Indeed, a piecewise monotone bounded function on a finite interval is of bounded variation.*

As a consequence of the above, we have

Corollary 4.1.37 *If f is a 2π-periodic function of class C^2 on $\mathbb{R}$, then the Fourier series of f converges normally on $\mathbb{R}$ and has sum f.*

Example 4.1.38

1. *Let $f :]-\pi, \pi] \mapsto \mathbb{R}$ be a periodic function, $T = 2\pi$ defined by $f(x) = x$.*

   a) *The discontinuities of f are the points of the form $x_k = (2k+1)\pi, k \in \mathbb{Z}$ and are of the first kind because $f(\pi^+) = \pi$ and $f(\pi^-) = -\pi$*

   b) *f is differentiable everywhere except at the points x_k. At these points we have:*

   $$\lim_{x\to\pi^-} \frac{f(x) - f(\pi)}{x - \pi} = 1 \text{ and } \lim_{x\to\pi^+} \frac{f(x) - f(\pi)}{x - \pi} = 1.$$

   f satisfies Dirichlet's conditions, so it is expandable in a Fourier series.

   f is odd, so $a_0 = a_n = 0$ and

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx = 2\frac{(-1)^{n+1}}{n}$$

   and consequently

   $$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

2. *Let $f : [-\pi, \pi] \mapsto \mathbb{R}$ be a function with period $T = 2\pi$, defined by $f(x) = |x|$.*

a) We have $|f(x)| \leq \pi$

b) $f_{|[-\pi,0]}$ *is decreasing and continuous and* $f_{[[0,\pi]}$ *is increasing and continuous.*

f *satisfies the conditions of Jordan's theorem, so it is expandable in a Fourier series. Moreover,* f *is even, which gives us* $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi} \pi|x|\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \begin{cases} 0 & if \quad n \quad even \\ -\frac{4}{\pi n^2} & if \quad n \quad odd \end{cases}$$

The Fourier series then converges to f *and we have* $f(x) = \dfrac{\pi}{2} - \displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$.

Since f *is continuous, the convergence is uniform.*

Finally, note that the equality $f(0) = 0$ *translates to* $\dfrac{\pi}{2} = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ *and consequently* $\dfrac{\pi^2}{8} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$.

One of the particularities of Fourier series is the calculation of the sums of certain infinite series.

4.1.11 Fourier Series of Functions with Arbitrary Period

Let f be a $2l$-periodic function, $l > 0$. To find the Fourier series of f, we make the following change of variable:

$$x = \frac{lt}{\pi}.$$

Then,

$$g(t) = f(\frac{lt}{\pi}),$$

is 2π-periodic. Indeed, for $k \in \mathbb{Z}$

$$g(t + 2k\pi) = f\left(\frac{l(t+2k\pi)}{\pi}\right) = f\left(\frac{lt}{\pi} + 2kl\right).$$

And since f is a $2l$-periodic function, we have

$$f\left(\frac{lt}{\pi} + 2kl\right) = f\left(\frac{lt}{\pi}\right) = g(t).$$

Thus,

$$g(t + 2k\pi) = g(t).$$

Under Dirichlet's conditions applied to the function g, we will have

$$g(t) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n \cos nt + b_n \sin nt)$$

where t is a point of continuity of g and where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt dt.$$

Therefore,

$$f\left(\frac{lt}{\pi}\right) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n \cos nt + b_n \sin nt), \tag{4.5}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\cos nt dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lt}{\pi}\right)\sin nt dt. \tag{4.6}$$

Now, $x = \frac{lt}{\pi}$ so

$$t = \frac{\pi}{l}x \quad \text{and} \quad dt = \frac{\pi}{l}dx$$

Moreover, if $t = \pm\pi$ then $x = \pm l$. Replacing t by its value in (4.5) and (4.6) , we obtain the Fourier series for a period $2l$, given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n \cos \frac{n\pi}{l}x + b_n \sin \frac{n\pi}{l}x),$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos \frac{n\pi}{l}x dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin \frac{n\pi}{l}x dx.$$

Example 4.1.39 *Consider the $2l$-periodic function f with $l > 0$, defined by*

$$f(x) = |x| \quad \textit{for} \quad -l \le x \le l.$$

Since f is even, the Fourier coefficients of f are:

$$\begin{cases} b_n = 0, \\ a_0 = \frac{2}{l}\int_0^l x dx = l, \\ a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi}{l} dx = \begin{cases} 0 & \textit{if } n \textit{ even} \\ \frac{-4l}{\pi^2 n^2} & \textit{if } n \textit{ odd} \end{cases} \end{cases}$$

Thus, the Fourier series for a period $2l$ is given by:

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{\cos \frac{\pi}{l}x}{1^2} + \frac{\cos \frac{3\pi}{l}x}{3^2} + ... + \frac{\cos \frac{(2k+1)\pi}{l}x}{(2k+1)^2} + ...\right].$$

4.1.12 Fourier Series in Complex Form

This representation of Fourier series in the form of the sum of sine and cosine functions is not unique. Indeed, we can also express the Fourier sum in exponential form, which simplifies the writing and calculations.

Let f be a 2π-periodic function such that

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n \cos nx + b_n \sin nx)$$

Therefore, according to subsection 4.1.8, the complex form of the Fourier series can be written in the following form:

$$f(x) = \sum_{n \in \mathbb{Z}} c_n \mathrm{e}^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} dx \quad \forall n \in \mathbb{Z}.$$

4.1.13 Fourier Series Expansion of Non-Periodic Functions

It is clear that the Fourier series expansion is performed on periodic functions. However, it is possible, in certain cases, to make such expansions for arbitrary functions.

Let $f : [a, b] \mapsto \mathbb{R}$ be a non-periodic function defined on the interval $[a, b]$. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a periodic function with period $T \geq b - a$ such that the restriction $g_{[a,b]} = f$. If g satisfies the Dirichlet's conditions, we will have:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

with a_n and b_n the Fourier coefficients associated with g. The sum of this series coincides everywhere with f in the interval $[a, b]$ except perhaps at the points of discontinuity of f.

Remark 4.1.40 *Let $f :]0, \ell[\mapsto \mathbb{R}$ be an arbitrary function, and $\ell > 0$. We assume that f can be extended to $]-\ell, 0[$ and that Dirichlet's or Jordan's conditions are satisfied. In this case, we have a choice for this extension. We can choose either an even extension or an odd extension to avoid lengthy calculations of the coefficients.*

Example 4.1.41 *Give a Fourier series of period 2π that coincides on $]0, \pi[$ with the function $f(x) = \mathrm{e}^x$.*
Here we only specify the interval where the Fourier series coincides with f, i.e. $]0, \pi[$. As the period of the Fourier series is 2π, there are then infinitely many answers; let us examine three different cases.

Let $\tilde{f}_i, i = 1, 2, 3$, be the extension of f to the entire real line. $\tilde{f}_i$ will be a function of period 2π that is exactly e^x for all x in $]0, \pi[$.

a) Let us choose an even extension and set:

$$\tilde{f}_1(x) = \begin{cases} \mathrm{e}^x & \text{if} \quad x \in]0, \pi[\\ \mathrm{e}^{-x} & \text{if} \quad x \in]-\pi, 0[\end{cases}.$$

It is easy to check that $\tilde{f}_1$ is an even function. Setting $\tilde{f}_1(0) = 1$ and $\tilde{f}_1(\pi) = \mathrm{e}^\pi$, we then have a continuous extension on $\mathbb{R}$. The graph of $\tilde{f}_1$ and the graph of the Fourier series will be identical.

Calculating the coefficients gives:

$$a_0 = \frac{2(\mathrm{e}^\pi - 1)}{\pi}, \quad a_n = 2\frac{(-1)^n \mathrm{e}^\pi - 1}{1 + n^2} \quad \textit{and} \quad b_n = 0.$$

We then have:

$$S_1(x) = \frac{e^{\pi} - 1}{\pi} + \sum_{n=1}^{\infty} 2\frac{(-1)^n e^{\pi} - 1}{n^2 + 1} \cos(nx) = \begin{cases} e^{x} & if \quad x \in [0, \pi] \\ e^{-x} & if \quad x \in [-\pi, 0] \end{cases}$$

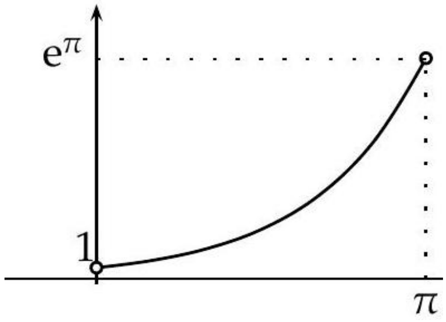


Figure 4.4: Graph of the function $f(x)$.

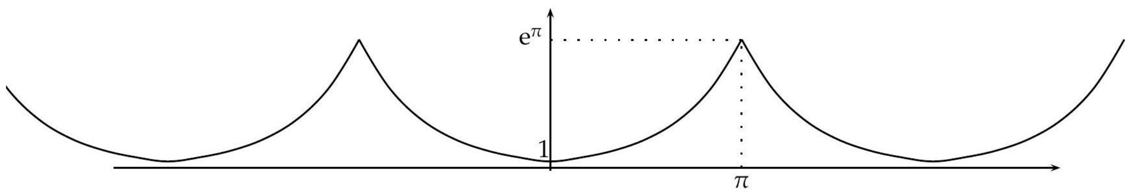


Figure 4.5: Graph of the function $S_1(x)$ identical to that of $\tilde{f}_1$.

b) Let us choose an odd extension and set:

$$\tilde{f}_2(x) = \begin{cases} e^{x} & if \quad x \in]0, \pi[\\ -e^{-x} & if \quad x \in]-\pi, 0[\end{cases}.$$

We note that $\tilde{f}_2$ is an odd function but is not continuous on $\mathbb{R}$. It is discontinuous at every point of the form $k\pi, k \in \mathbb{Z}$.

Calculating the coefficients gives:

$$a_n = 0, \forall n \in \mathbb{N}, \quad b_n = \frac{2n\,(1 - (-1)^n e^{\pi})}{\pi\,(1 + n^2)}.$$

We then have:

$$S_2(x) = \sum_{n=1}^{\infty} \frac{2n\,(1 - (-1)^n e^{\pi})}{\pi\,(1 + n^2)} \sin(nx) = \begin{cases} e^{x} \; if & x \in]0, \pi[\\ -e^{-x} \; if & x \in]-\pi, 0[\\ 0 \; if & x = 0 \; ou \; x = \pm\pi \end{cases}$$

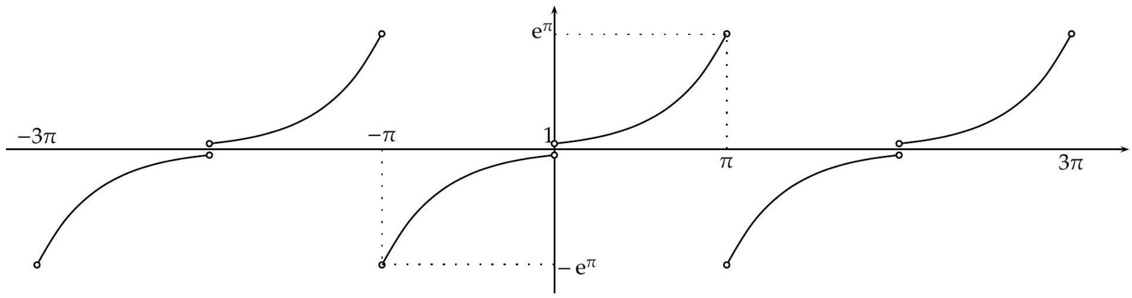


Figure 4.6: Graph of the function $\tilde{f}_2(x)$.

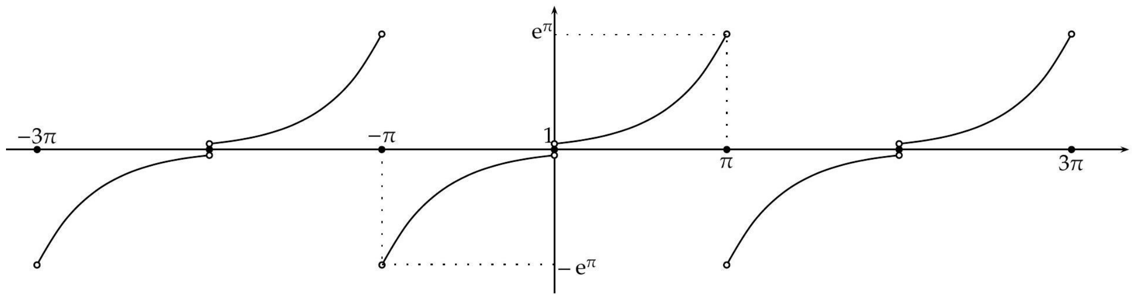


Figure 4.7: Graph of the series $S_2(x)$.

c) *Let us choose an extension that is neither even nor odd and set:* $\tilde{f}_3(x) = \mathrm{e}^x$ *if* $x \in]-\pi, \pi[$. *We note that* f *is discontinuous at every point of the form* $\pi + 2k\pi, k \in \mathbb{Z}$. *We have the final result:*

$$S_3(x) = \frac{\mathrm{e}^\pi - \mathrm{e}^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}(\cos(nx) - n\sin(nx))\right) = \begin{cases} \mathrm{e}^x & \text{if } x \in]-\pi, \pi[\\ \dfrac{\mathrm{e}^\pi + \mathrm{e}^{-\pi}}{2} & \text{if } x = \pm\pi \end{cases}$$

We have obtained three different series that are exactly equal to e^x *on the interval* $]0, \pi[$. *We could have chosen other extensions and obtained other series.*

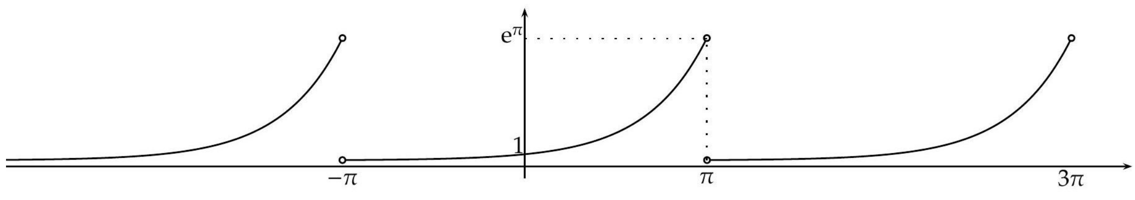


Figure 4.8: Graph of the function $\tilde{f}_3(x)$.

Remark 4.1.42 *If we wanted a Fourier series with period* π, *then there is only one that coincides with* f *on* $]0, \pi[$.

We find;

$$S_4(x) = \frac{2(\mathrm{e}^\pi - 1)}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{4n^2-1}(\cos(2nx) - 2n\sin(2nx))\right) = \begin{cases} \mathrm{e}^x & \text{if} \quad x \in]0, \pi[\\ \dfrac{1+\mathrm{e}^\pi}{2} & \text{if} \quad x = 0 \text{ or } x = \pi \end{cases}$$

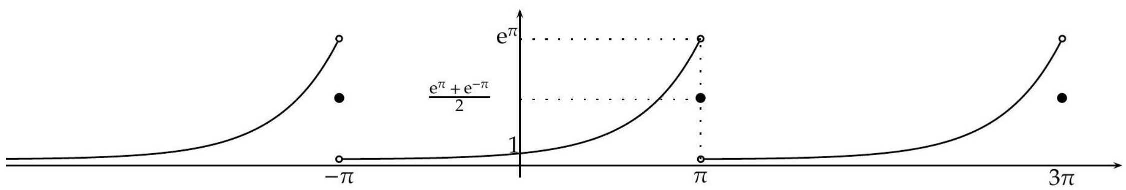


Figure 4.9: Graph of the series $S_3(x)$.

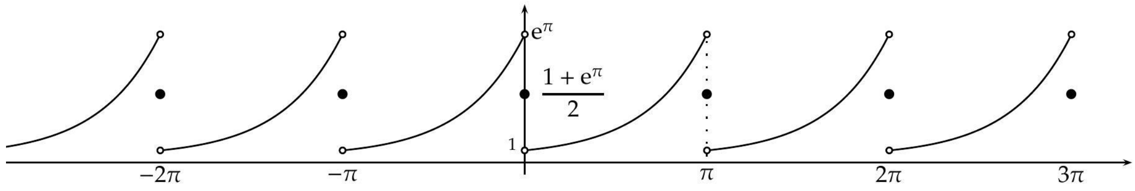


Figure 4.10: graph of the series $S_4(x)$.

4.1.14 Bessel's Inequality

The following proposition is called Bessel's inequality. It follows from the projection theorem on a finite-dimensional space.

Proposition 4.1.43 *Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a 2π-periodic function, Riemann integrable on $[0, 2\pi]$, with Fourier coefficients a_n and b_n. Then the infinite series $\sum_{n\geq 1} |a_n|^2$ and $\sum_{n\geq 1} |b_n|^2$ converge, and we have*

$$\frac{|a_0|^2}{2} + \sum_{n\geq 1}(|a_n|^2 + |b_n|^2) \leq \frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx.$$

In the case of complex coefficients, Bessel's inequality is written as:

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx.$$

Proof: Let us prove the complex form, which implies the real form via the relations

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}\ (n > 0), \quad c_{-n} = \overline{c_n}\ (n > 0),$$

and the identity $|c_n|^2 + |c_{-n}|^2 = \dfrac{|a_n|^2 + |b_n|^2}{2}$ for $n \geq 1$.

1. Setup. Consider the N-th partial sum of the Fourier exponential series:

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt.$$

Define the approximation error

$$E_N = \frac{1}{2\pi}\int_0^{2\pi} \left|f(x) - S_N(x)\right|^2 dx.$$

2. Expansion of the error. Since f and S_N are 2π-periodic, we may compute over any interval of length 2π. Expanding the square gives

$$E_N = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx - \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{S_N(x)}\,dx$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}S_N(x)\,dx + \frac{1}{2\pi}\int_0^{2\pi} |S_N(x)|^2\,dx.$$

3. Orthogonality relations. For the mixed terms, use the definition of c_n and the orthogonality

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-k)x}\,dx = \delta_{nk} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}.$$

Indeed,

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{S_N(x)}\,dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sum_{n=-N}^{N}\overline{c_n}e^{-inx}\,dx = \sum_{n=-N}^{N}\overline{c_n}\cdot\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \sum_{n=-N}^{N}|c_n|^2.$$

Similarly,

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{f(x)}S_N(x)\,dx = \sum_{n=-N}^{N}|c_n|^2.$$

For the purely trigonometric term,

$$\frac{1}{2\pi}\int_0^{2\pi} |S_N(x)|^2\,dx = \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-N}^{N}\sum_{m=-N}^{N} c_n\overline{c_m}e^{i(n-m)x}\,dx = \sum_{n=-N}^{N}|c_n|^2,$$

again by orthogonality.

4. Expression of the error. Substituting these results into E_N yields

$$E_N = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx - \sum_{n=-N}^{N}|c_n|^2.$$

5. Positivity of E_N. Since E_N is the integral of a non-negative function,

$$E_N \geq 0 \quad \text{for all } N.$$

Hence

$$\sum_{n=-N}^{N}|c_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx.$$

Letting $N \to \infty$, we obtain Bessel's inequality in complex form:

$$\sum_{n=-\infty}^{+\infty}|c_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx.$$

6. Real form. Using the relations between (a_n, b_n) and (c_n), we have

$$|c_0|^2 = \frac{|a_0|^2}{4}, \qquad |c_n|^2 + |c_{-n}|^2 = \frac{|a_n|^2 + |b_n|^2}{2} \quad (n \geq 1).$$

Therefore

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{2}.$$

Multiplying Bessel's complex inequality by 2 gives

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} \big(|a_n|^2 + |b_n|^2\big) \leq \frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 \, dx.$$

7. Convergence of the series. Since the partial sums $\sum_{n=-N}^{N} |c_n|^2$ are bounded above by the fixed constant $\frac{1}{2\pi} \int_0^{2\pi} |f|^2$, and they are increasing with N, they converge by the monotone convergence theorem. Consequently, $\sum_{n=1}^{\infty} |a_n|^2$ and $\sum_{n=1}^{\infty} |b_n|^2$ also converge. □

4.1.15 Parseval's Identity

The following result, which we will admit, is due to Parseval and is of great importance in signal processing. Parseval's formula indicates that the total energy of a periodic signal is obtained by summing the energy of the different harmonics of the signal. It is an improvement on Bessel's inequality.

Theorem 4.1.44 (Parseval's Identity) *Let $f : \mathbb{R} \mapsto \mathbb{C}$ be a 2π-periodic function, Riemann integrable on $[0, 2\pi]$, with Fourier coefficients a_n and b_n. Then the infinite series $\sum_{n\geq 1} |a_n|^2$ and $\sum_{n\geq 1} |b_n|^2$ converge, and we have*

$$\frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n\geq 1} (|a_n|^2 + |b_n|^2).$$

In the case of complex coefficients, Parseval's identity is written as:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = |c_0|^2 + \sum_{n=1}^{+\infty} (|c_n|^2 + |c_{-n}|^2).$$

Proof: For the proof, see [7]. □

Remark 4.1.45 *If f is T-periodic and satisfies the hypotheses of the previous theorem, Parseval's identity can be written as:*

$$\frac{2}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n\geq 1} (|a_n|^2 + |b_n|^2).$$

Remark 4.1.46 *Parseval's identity allows us to assert that there exist trigonometric series that are not the Fourier series of any periodic and integrable function. For example, the trigonometric series $\sum_{n\geq 1} \frac{1}{\sqrt{n}} \sin nx$. According to Abel's rule, this series converges pointwise on $\mathbb{R}\backslash\{2\pi\mathbb{Z}\}$. Moreover, for $x = 2\pi\mathbb{Z}$, this series is the null series, so it converges pointwise on $\mathbb{R}$. If this* *trigonometric series were the Fourier series of a function f that is 2π-periodic and Riemann integrable on $[0, 2\pi]$, we would then have*

$$\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=1}^{+\infty} \frac{1}{n}.$$

This equality leads to a contradiction because, since f is square integrable on $[0, 2\pi]$, the integral $\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx$ is well-defined, whereas the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ diverges.

Remark 4.1.47

1. *If f has period 2π, we have:*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi}\int_0^{2\pi} f^2(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

2.

$$f \text{ even function } \Longrightarrow f^2 \text{ even function } \Longrightarrow \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{2}{\pi}\int_0^{\pi} f^2(x)dx$$

$$f \text{ odd function } \Longrightarrow f^2 \text{ even function } \Longrightarrow \sum_{n=1}^{\infty} b_n^2 = \frac{2}{\pi}\int_0^{\pi} f^2(x)dx$$

4.1.16 Applications

Example 4.1.48 *Let f be a 2π-periodic function such that:*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in]0, \pi[\\ -1 & \text{if} \quad x \in]-\pi, 0[\end{cases}$$

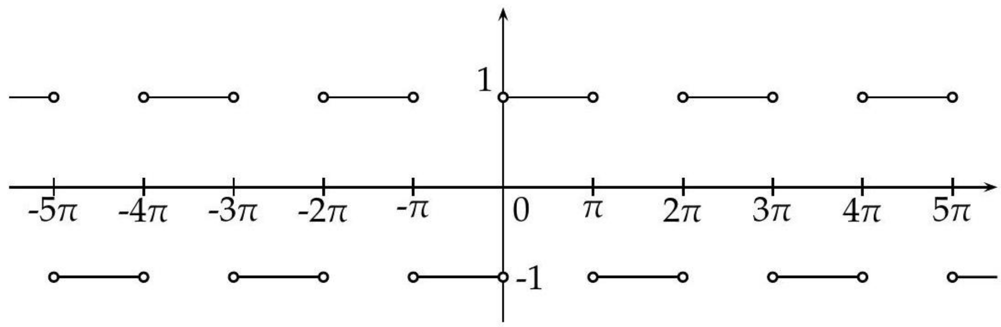


Figure 4.11: graph of the function $f(x)$.

Since f is an odd function $\Longrightarrow a_n = 0, \forall n \in \mathbb{N}$.

We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nt)dt = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \textit{if} \quad n \quad \textit{is even} \\ \\ \frac{4}{n\pi} & \textit{if} \quad n \quad \textit{is odd} \end{cases}$$

The Fourier series associated with it is:

$$S(x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = \begin{cases} 1 & \textit{if} \quad x \in]0,\pi[\\ \\ 0 & \textit{if} \quad x = 0 \ \textit{or}\ x = \pi \end{cases}$$

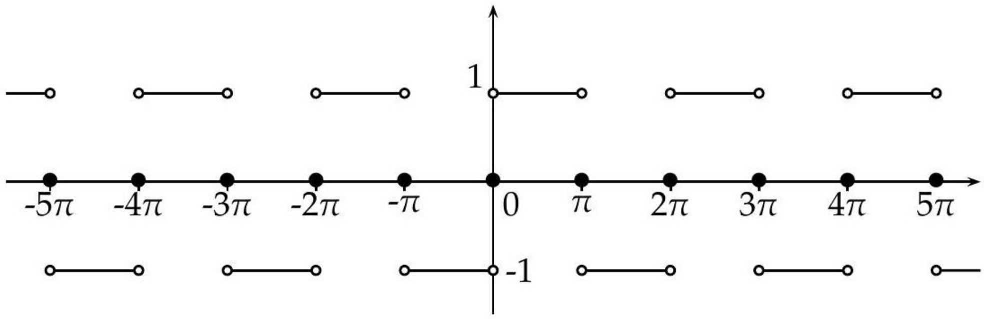


Figure 4.12: graph of the function $S(x)$.

Remark 4.1.49 *For $x = \pi/2$ we have*

$$S(\pi/2) = 1 = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi/2}{2n+1} = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

We obtain:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Applying Parseval's identity:

$$\frac{2}{\pi}\int_0^{\pi} f^2(t)dt = 2 = \sum_{n=0}^{\infty} \frac{16}{\pi^2}\cdot\frac{1}{(2n+1)^2}$$

and we therefore obtain:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Remark 4.1.50 *Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ be a convergent series according to the Riemann criterion. By separating the even and odd terms, we have: $S = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} (\star)$. As $\sum_{n=1}^{\infty} \frac{1}{(2n)^2} =$*

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{S}{4}$; *substituting in the equality* $(\star)$ *we have:*

$$S=\frac{S}{4}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\Longleftrightarrow\frac{3S}{4}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}\Longleftrightarrow S=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

The complex method:

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-int}f(t)dt=\frac{1}{2\pi}\left(\int_{-\pi}^{0}-\mathrm{e}^{-int}dt+\int_{0}^{\pi}\mathrm{e}^{-int}dt\right)=\frac{1}{2\pi}\left[\left(\frac{\mathrm{e}^{-int}}{in}\right)_{-\pi}^{0}+\left(\frac{\mathrm{e}^{-int}}{-in}\right)_{0}^{\pi}\right]$$
$$=-i\frac{1-(-1)^n}{\pi n}=1/2\,(a_n-ib_n)$$

Example 4.1.51 *Let* f *be a* 2π*-periodic function such that:*

$$f(x)=|x|\quad \text{if}\quad x\in[-\pi,\pi]$$

Since f *is an even function* $\Longrightarrow b_n=0,\forall n\in\mathbb{N}$
We have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|t|dt=\frac{2}{\pi}\int_{0}^{\pi}tdt=\frac{2}{\pi}\left(\frac{t^2}{2}\right)_0^{\pi}=\pi$$

and

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|t|\cos ntdt=\frac{2}{\pi}\int_{0}^{\pi}t\cos ntdt=\frac{2}{\pi}\left(\left[\frac{t\sin nt}{n}\right]_0^{\pi}-\int_0^{\pi}\frac{\sin nt}{n}dt\right)$$
$$=\frac{-2}{n\pi}\int_0^{\pi}\sin ntdt=\frac{2}{n\pi}\left(\frac{\cos nt}{n}\right)_0^{\pi}=\frac{2}{\pi n^2}\left((-1)^n-1\right)=\begin{cases}0 & \text{if } n \text{ is even}\\ \frac{-4}{\pi n^2} & \text{if } n \text{ is odd}\end{cases}$$

The associated series is therefore:

$$S(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$$

Since f *is a continuous function on* $\mathbb{R}$*, and has right and left derivatives everywhere, then* $f(x)=S(x),\forall x\in\mathbb{R}$.

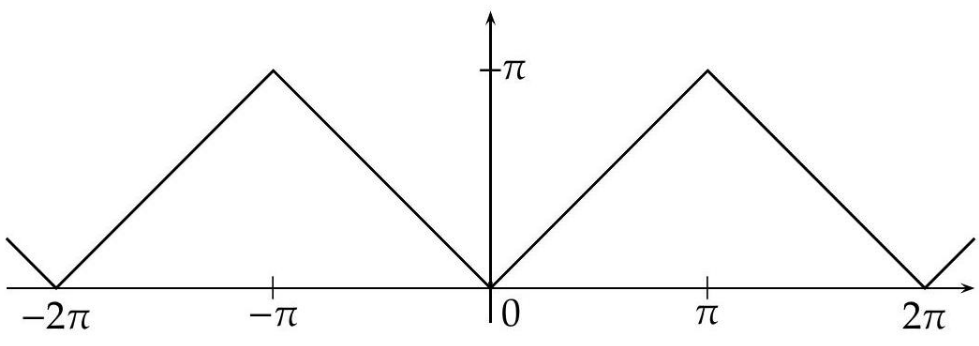


Figure 4.13: graph of the function $f(x)$ and that of $S(x)$.

Remark 4.1.52

1. *for* $x = 0$ *we have* $f(0) = 0$ *and since* $f(0) = S(0)$ *we obtain:*

$$S(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Longleftrightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Parseval's identity gives:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\cdot\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}$$

Therefore

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

2. *By writing*

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{(2n)^4} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}.$$

We then deduce:

$$\frac{15S}{16} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

or

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

4.2 Exercises of the Chapter

Exercise 4.2.1 *Calculate the trigonometric Fourier series of the* 2π*-periodic function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = \pi - |x|$ *on* $(-\pi, \pi]$*. Does the series converge to* f*?*

Correction 4.2.1 *It is easy to see that the function* f *is even, so the coefficients* b_n *are all zero, and*

$$a_n(f) = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(nt)\,dt = 2\int_0^{\pi}\cos(nt)\,dt - \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \begin{cases} \dfrac{2}{\pi n^2}\left(1-(-1)^n\right) & \text{if } n \neq 0, \\ \pi & \text{if } n = 0. \end{cases}$$

Thus, we have:

$$SF(f)(t) = \frac{\pi}{2} + \sum_{k\geq 1}\frac{4}{\pi(2k+1)^2}\cos((2k+1)t).$$

Since the function f *is continuous on* $\mathbb{R}$*, Dirichlet's theorem shows that the series converges to* f *at every point of* $\mathbb{R}$*.*

Exercise 4.2.2 *Calculate the Fourier series, in trigonometric form, of the* 2π*-periodic function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = x^2$ *on* $[0, 2\pi)$*. Does the series converge to* f*?*

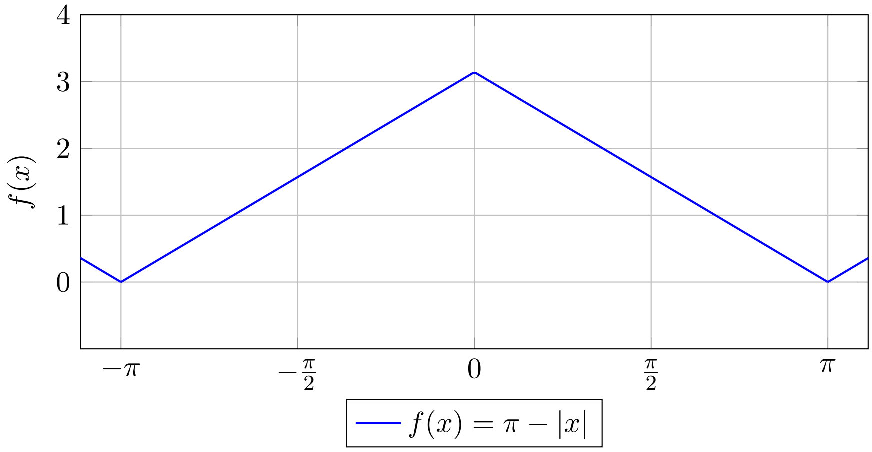


Figure 4.14: 2π-periodic function $f(x) = \pi - |x|$ on $(-\pi, \pi]$ and its periodic extension

Correction 4.2.2 *The function f is neither even nor odd. We will calculate its trigonometric Fourier coefficients.*

On the one hand,

$$a_0(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{3},$$

and on the other hand, for $n \geq 1$,

$$\begin{aligned} a_n(f) &= \frac{1}{\pi}\int_0^{2\pi} t^2\cos(nt)\,dt \\ &= \frac{1}{\pi}\left[\frac{t^2\sin(nt)}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi}\frac{2t\sin(nt)}{n}\,dt \\ &= -\frac{2}{\pi}\left\{\left[-\frac{t\cos(nt)}{n^2}\right]_0^{2\pi} + \int_0^{2\pi}\frac{\cos(nt)}{n^2}\,dt\right\} \\ &= -\frac{2}{\pi}\left\{-\frac{2\pi}{n^2} + \left[\frac{\sin(nt)}{n^3}\right]_0^{2\pi}\right\} \\ &= \frac{4}{n^2} \end{aligned}$$

and

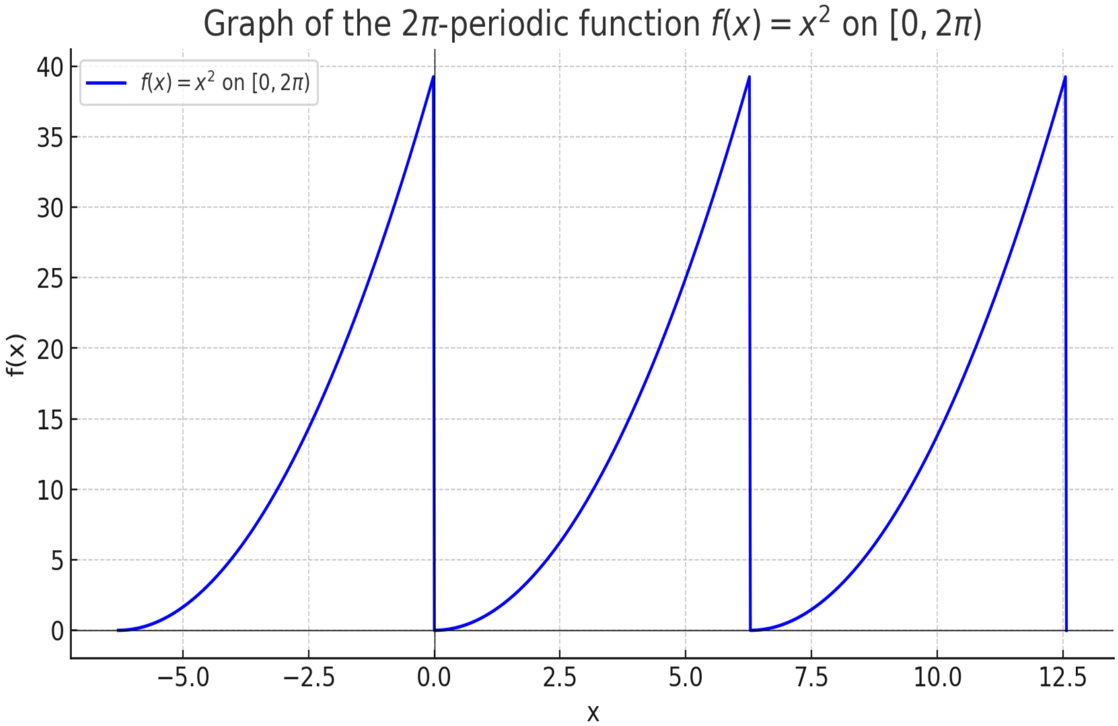
$$\begin{aligned} b_n(f) &= \frac{1}{\pi}\int_0^{2\pi} t^2\sin(nt)\,dt \\ &= \frac{1}{\pi}\left\{\left[-\frac{t^2\cos(nt)}{n}\right]_0^{2\pi} + \int_0^{2\pi}\frac{2t\cos(nt)}{n}\,dt\right\} \\ &= \frac{1}{\pi}\left\{-\frac{4\pi^2}{n} + \left[\frac{2t\sin(nt)}{n^2}\right]_0^{2\pi} - 2\int_0^{2\pi}\frac{\sin(nt)}{n^2}\,dt\right\} \\ &= -\frac{4\pi}{n} + \frac{2}{\pi}\left[\frac{\cos(nt)}{n^3}\right]_0^{2\pi} \\ &= -\frac{4\pi}{n}. \end{aligned}$$

Thus, we have:

$$SF(f)(t) = \frac{4\pi^2}{3} + 4\sum_{n\geq 1}\left(\frac{\cos(nt)}{n^2} - \frac{\pi\sin(nt)}{n}\right).$$

The function f satisfies the hypotheses of Dirichlet's theorem, and the Fourier series $SF(f)$ converges for all t to

$$\frac{f(t_+) + f(t_-)}{2} = \begin{cases} f(t) & \text{if } t \neq 2\pi\mathbb{Z}, \\ 2\pi^2 & \text{if } t = 2\pi\mathbb{Z}. \end{cases}$$



Exercise 4.2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic, odd function, such that*

$$f(x) = \begin{cases} 1 & \text{if } x \in]0, \pi[\\ 0 & \text{if } x = \pi. \end{cases}$$

1. *Calculate the trigonometric Fourier coefficients of f.*

2. *Study the convergence (pointwise, uniform) of the Fourier series of f.*

3. *Deduce the values of the sums*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}, \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

Correction 4.2.3

1. *Since the function f is odd, $a_n = 0$ for all $n \in \mathbb{N}$. For $n \geq 1$,*

$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} \sin(nt)\,dt = \left[-\frac{\cos(nt)}{n}\right]_0^{\pi} = \frac{2}{\pi}\frac{1-(-1)^n}{n} = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

The trigonometric Fourier series of f is therefore given by

$$SF(f)(t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)t).$$

2. *The function f satisfies the hypotheses of Dirichlet's theorem, so the series $SF(f)$ converges at every t to*

$$\frac{f(t^+) + f(t^-)}{2} = f(t).$$

3. *For $t = \pi/2$, we have:*

$$\sin((2k+1)t) = \sin\left(\frac{\pi}{2} + k\pi\right) = (-1)^k,$$

so

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} f\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

Since f is odd, Parseval's equality gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \frac{1}{2}\sum_{n=1}^{\infty} |b_n(f)|^2 = \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

so

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Next, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Finally,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} - \frac{1}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{12}.$$

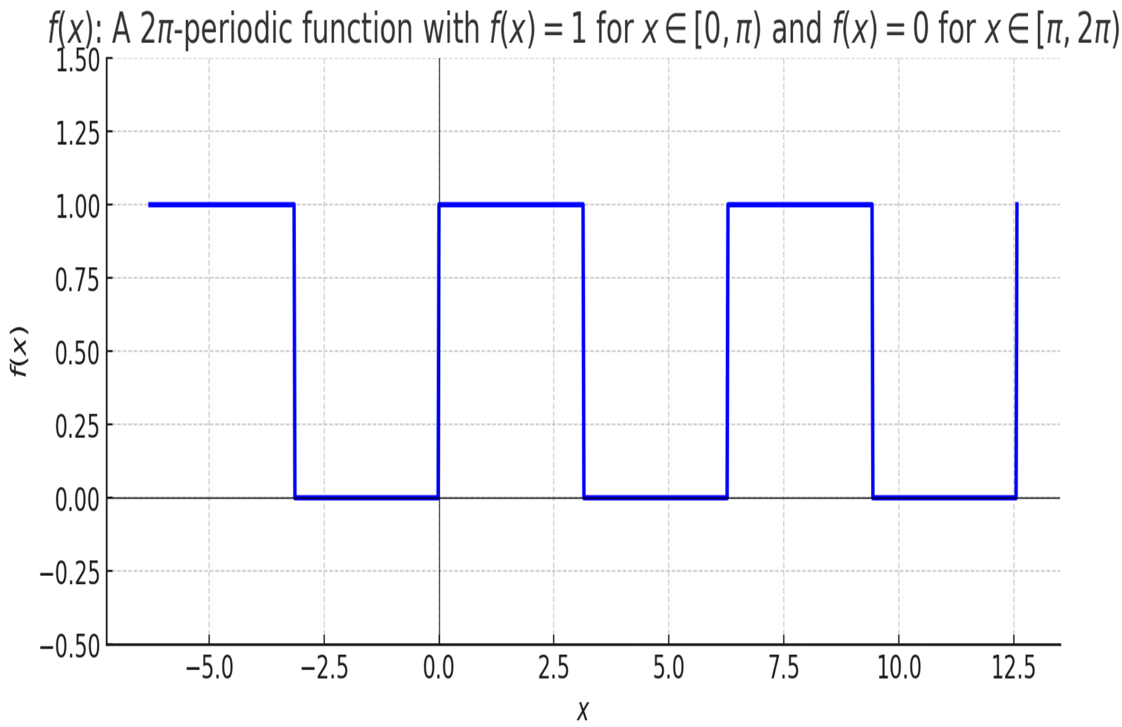
Exercise 4.2.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function such that $f(x) = e^x$ for all $x \in]-\pi, \pi]$.*

1. *Compute the exponential Fourier coefficients of the function f.*

2. *Study the convergence (pointwise and uniform) of the Fourier series of f.*

3. *Deduce the values of the sums*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}, \quad \sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$



Correction 4.2.4

1. *We have:*

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t e^{-int}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)t}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\left[\frac{e^{(1-in)t}}{1-in}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left(\frac{e^{(1-in)\pi}-e^{-(1-in)\pi}}{1-in}\right) \\ &= \frac{1}{2\pi}\left((-1)^n\frac{e^{\pi}-e^{-\pi}}{1-in}\right) \\ &= \frac{\sinh(\pi)}{\pi}\frac{(-1)^n}{1-in}. \end{aligned}$$

2. *It is easy to verify that the hypotheses of Dirichlet's theorem are satisfied. It follows that:*

$$SF(f)(t) = c_0(f) + \sum_{n\geq 1}\left(c_n(f)e^{int} + c_{-n}(f)e^{-int}\right)$$

converges to $f(t)$ *if* $t \in]-\pi,\pi[$, *and to* $(f(\pi^+)+f(\pi^-))/2 = \cosh(\pi)$ *if* $t=\pi$. *In other words:*

$$\frac{\sinh(\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{1-in}e^{int}=\begin{cases}e^t & \text{if } t\in]-\pi,\pi[,\\ \cosh(\pi) & \text{if } t=\pi.\end{cases}$$

Since the sum function is not continuous, the convergence cannot be uniform.

3. *For $t=0$, we obtain:*

$$1=\frac{\sinh(\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{1-in},$$

which gives:

$$\frac{\pi}{\sinh(\pi)}=1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{1-in}+\frac{1}{1+in}\right)=-1+\sum_{n=0}^{\infty}\frac{2(-1)^n}{1+n^2},$$

hence:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1+n^2}=\frac{1}{2}\left(\frac{\pi}{\sinh(\pi)}+1\right).$$

For $t=\pi$, we obtain:

$$\cosh(\pi)=\frac{\sinh(\pi)}{\pi}\sum_{n\in\mathbb{Z}}\frac{1}{1-in},$$

which gives:

$$\frac{\pi}{\operatorname{th}(\pi)}=\sum_{n\in\mathbb{Z}}\frac{1}{1-in}=-1+\sum_{n=0}^{\infty}\frac{2}{1+n^2},$$

hence:

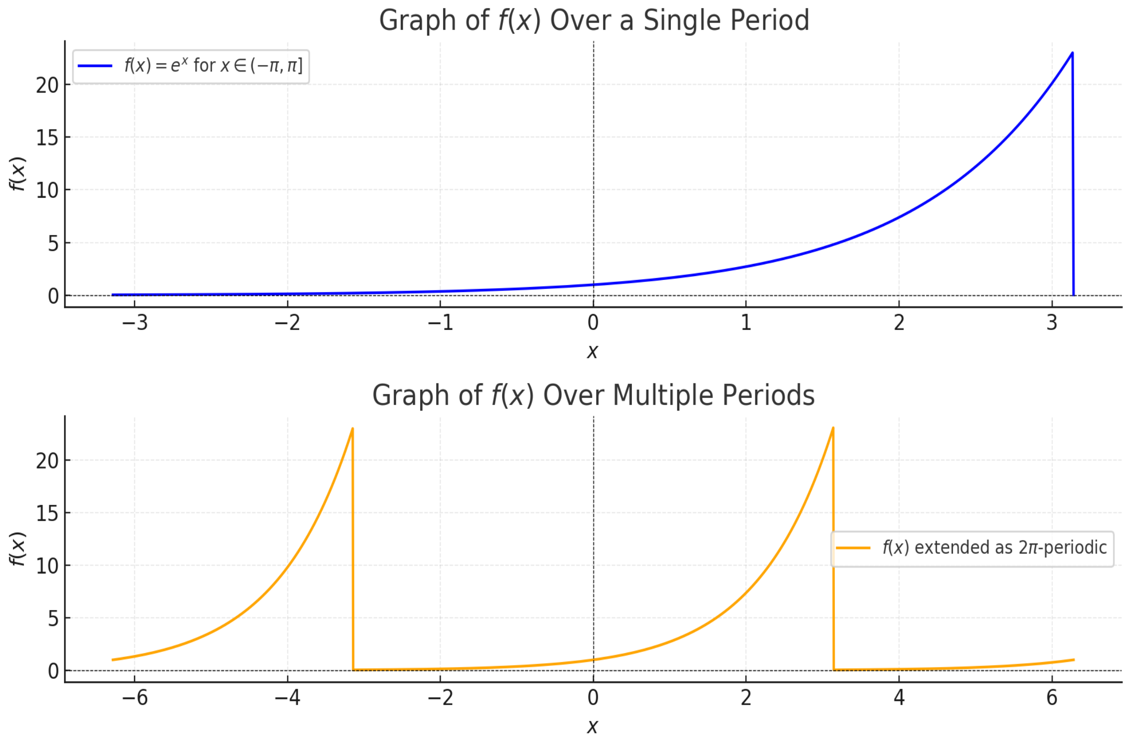
$$\sum_{n=0}^{\infty}\frac{1}{1+n^2}=\frac{1}{2}\left(\frac{\pi}{\operatorname{th}(\pi)}+1\right).$$

Exercise 4.2.5 *Let $f:\mathbb{R}\to\mathbb{R}$ be the 2π-periodic function defined by:*

$$f(x)=(x-\pi)^2,\quad x\in[0,2\pi[.$$

1. *Compute the trigonometric Fourier coefficients of the function f.*
2. *Study the convergence of the Fourier series of f.*
3. *Deduce the sums of the series.*

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},\quad \sum_{n=1}^{\infty}\frac{1}{n^2}.$$



4. *Using Parseval's formula, show the equality:*

$$\sum_{n=0}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Correction 4.2.5

1. *We note that f is an even function, so $b_n(f) = 0$ for all n. Furthermore:*

$$a_0(f) = \frac{1}{\pi} \int_0^{2\pi} (x-\pi)^2 \,\mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} y^2 \,\mathrm{d}y = \frac{1}{\pi} \left[\frac{y^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

and, for all $n \geq 1$:

$$\begin{aligned} a_n(f) &= \frac{1}{\pi} \int_0^{2\pi} (x-\pi)^2 \cos(nx) \,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} y^2 \cos(ny + n\pi) \,\mathrm{d}y \\ &= \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} y^2 \cos(ny) \,\mathrm{d}y \\ &= \frac{(-1)^n}{\pi} \cdot \frac{4}{n^2} (-1)^n \pi \\ &= \frac{4}{n^2}, \end{aligned}$$

where two integration's by parts were performed. The Fourier series of f is therefore:

$$SF(f)(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$

2. *The function f is of class $\mathscr{C}^1(\mathbb{R})$. Thus, Dirichlet's theorem implies that for all $x \in \mathbb{R}$, $SF(f)(x) = f(x)$.*

3. *From the previous question:*

$$\pi^2 = f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \textit{and} \quad 0 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

We deduce that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

4. *Applying Parseval's equality:*

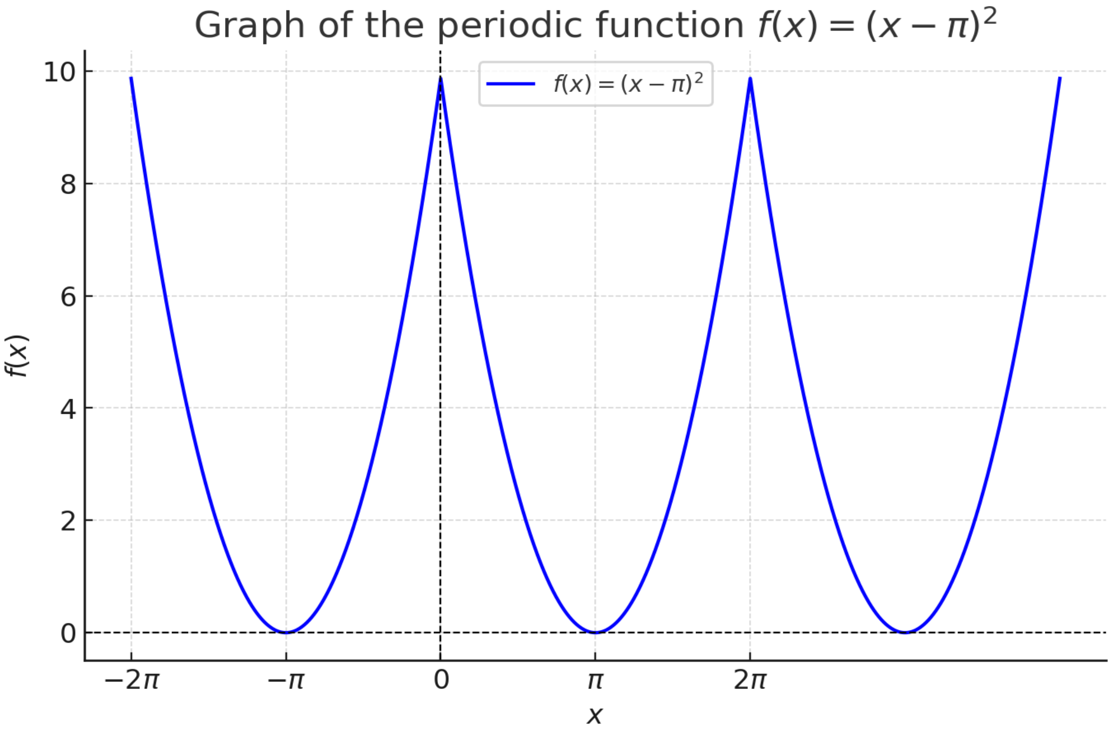
$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,dx,$$

we obtain:

$$\frac{2\pi^4}{5} = \frac{1}{\pi}\int_0^{2\pi} (x-\pi)^4\,dx = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}.$$

Thus, we conclude:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$



Graph of the periodic function $f(x) = (x - \pi)^2$
$f(x) = (x - \pi)^2$
10
8
6
4
2
0
$f(x)$
-2π
$-\pi$
0
π
2π
x

Chapter 5

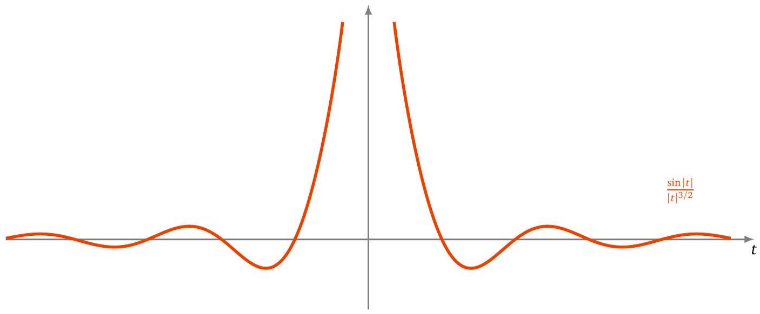
Improper Integral

5.1 Definitions and First Properties

Most of the integrals you will encounter are not areas of bounded domains in the plane. Here we will learn how to calculate integrals of unbounded domains, either because the interval of integration is infinite (going up to $+\infty$ or $-\infty$), or because the function to be integrated tends to infinity at the bounds of the interval. Indeed, for any closed bounded interval $I = [a, b]$ with a and b real, and for any function f that is continuous or piecewise continuous on I, it is possible to define the Riemann integral $\int_a^b f(t)dt$ as the limit of Riemann sums. It is a matter of extending this definition to functions defined on the open interval $]a, b[$ bounded or not.

5.1.1 Singular Points

Consider for example the function f which associates to $t \in]-\infty, 0[\cup]0, +\infty[$ the value $f(t) = \dfrac{\sin|t|}{|t|^{\frac{3}{2}}}$. How can we give meaning to the integral of f on $\mathbb{R}$?



- First, we identify the singular points, either $+\infty$ or $-\infty$ on the one hand, and on the other hand the point or points in the neighborhood of which the function is not bounded ($t = 0$ in our example).
- We then divide each interval of integration into as many intervals as necessary so that each of them contains only one singular point, placed at one of the two bounds.
- We want a definition that respects Chasles' relation. Thus, the integral over the complete interval is the sum of the integrals over the intervals of the subdivision.
- In the example of the function $f(t) = \dfrac{\sin|t|}{|t|^{\frac{3}{2}}}$ above, it is necessary to divide the two intervals of definition $]-\infty, 0[$ and $]0, +\infty[$ into 4 subintervals: 2 to isolate $-\infty$ and $+\infty$, and 2 others for the singular point 0.

- We can write for this example:

$$\int_{-\infty}^{+\infty} f(t)\mathrm{d}t = \int_{-\infty}^{-1} f(t)\mathrm{d}t + \int_{-1}^{0} f(t)\mathrm{d}t + \int_{0}^{1} f(t)\mathrm{d}t + \int_{1}^{+\infty} f(t)\mathrm{d}t$$

- The only goal is to isolate the difficulties: the choices of -1 and 1 as subdivision points are arbitrary (for example -3 and 10 would have been just as good).

5.1.2 Convergence/Divergence

Through this subdivision, and by changing the variable $t \mapsto -t$, we are reduced to integrals of two types.

1. Integral over $[a, +\infty[$.

2. Integral over $]a, b]$, with the function unbounded at a.

We must therefore define an integral, called an improper integral, in these two cases.

Definition 5.1.1

1. *Let f be a continuous function on $[a, +\infty[$ where $a \in \mathbb{R}$. We say that the integral $\int_a^{+\infty} f(t)\mathrm{d}t$ converges if the limit, as x tends to $+\infty$, of the primitive $\int_a^x f(t)\mathrm{d}t$ exists and is finite. If this is the case, we set:*

$$\int_a^{+\infty} f(t)\mathrm{d}t = \lim_{x \to +\infty} \int_a^x f(t)\mathrm{d}t$$

Otherwise, we say that the integral diverges.

2. *Let f be a continuous function on $]a, b]$. We say that the integral $\int_a^b f(t)\mathrm{d}t$ converges if the right-hand limit, as x tends to a, of $\int_x^b f(t)\mathrm{d}t$ exists and is finite. If this is the case, we set:*

$$\int_a^b f(t)\mathrm{d}t = \lim_{x \to a^+} \int_x^b f(t)\mathrm{d}t$$

Otherwise, we say that the integral diverges.

Remark 5.1.2 *Convergence is therefore equivalent to a finite limit. Divergence means either that there is no limit, or that the limit is infinite.*

- *Observe that the second definition is consistent with the integral of a function that would be continuous on $[a, b]$ as a whole (instead of $]a, b]$). We know that the primitive $\int_x^b f(t)\mathrm{d}t$ is a continuous function. Consequently, the usual integral $\int_a^b f(t)\mathrm{d}t$ is also the limit of $\int_x^b f(t)\mathrm{d}t$ (as $x \to a^+$). In this case, the two integrals coincide.*

Examples 5.1.3 *When we can calculate a primitive $F(x)$ of the function to be integrated (for example $F(x) = \int_a^x f(t)\mathrm{d}t$), the study of convergence is reduced to a calculation of the limit of $F(x)$. Here are several examples.*

1. *The integral*

$$\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t \quad \textit{converges.}$$

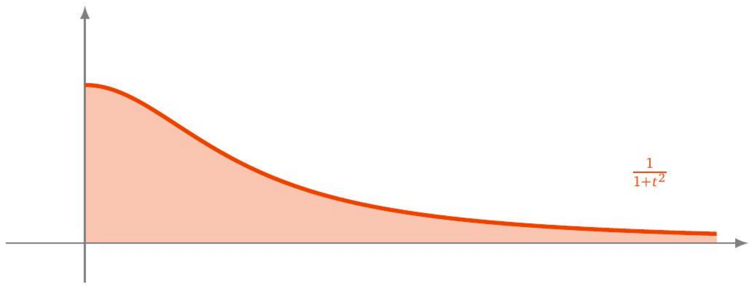
Indeed,

$$\int_0^{x} \frac{1}{1+t^2}\,\mathrm{d}t = [\arctan t]_0^x = \arctan x \quad \textit{and} \quad \lim_{x\to+\infty} \arctan x = \frac{\pi}{2}.$$

We can write:

$$\int_0^{+\infty} \frac{1}{1+t^2}\,\mathrm{d}t = [\arctan t]_0^{+\infty} = \frac{\pi}{2}$$

provided we remember that $[\arctan t]_0^{+\infty}$ denotes a limit at $+\infty$.



This proves that the area under the curve is not bounded, but its area is finite!

2. *However, the integral*

$$\int_0^{+\infty} \frac{1}{1+t}\,\mathrm{d}t \quad \textit{diverges.}$$

Indeed,

$$\int_0^{x} \frac{1}{1+t}\,\mathrm{d}t = [\ln(1+t)]_0^x = \ln(1+x) \quad \textit{and} \quad \lim_{x\to+\infty} \ln(1+x) = +\infty.$$

3. *The integral*

$$\int_0^{1} \ln t\ \mathrm{d}t \quad \textit{converges.}$$

Indeed, we can write:

$$\int_x^{1} \ln t\ \mathrm{d}t = [t\ln t - t]_x^1 = x - x\ln x - 1 \quad \textit{and} \quad \lim_{x\to 0^+} (x - x\ln x - 1) = -1$$

$$\int_0^{1} \ln t\ \mathrm{d}t = [t\ln t - t]_0^1 = -1$$

4. However, the integral

$$\int_0^1 \frac{1}{t}\,\mathrm{d}t \quad \textit{diverges.}$$

Indeed,

$$\int_x^1 \frac{1}{t}\,\mathrm{d}t = [\ln t]_x^1 = -\ln x \quad \textit{and} \quad \lim_{x\to 0^+} -\ln x = +\infty$$

5.1.3 Chasles' Relation

When it converges, this new integral satisfies the same properties as the usual Riemann integral, starting with Chasles' relation:

Proposition 5.1.4 (Chasles' Relation) *Let $f : [a, +\infty[\to \mathbb{R}$ be a continuous function and let $a' \in [a, +\infty[$. Then the improper integrals $\int_a^{+\infty} f(t)\mathrm{d}t$ and $\int_{a'}^{+\infty} f(t)\mathrm{d}t$ are of the same nature. If they converge, then*

$$\int_a^{+\infty} f(t)\mathrm{d}t = \int_a^{a'} f(t)\mathrm{d}t + \int_{a'}^{+\infty} f(t)\mathrm{d}t$$

"Being of the same nature" means that both integrals are convergent at the same time or divergent at the same time.

Chasles' relation therefore implies that convergence does not depend on the behavior of the function over bounded intervals, but only on its behavior in the neighborhood of $+\infty$.

Proof: The proof follows from Chasles' relation for the usual integrals, with $a \leqslant a' \leqslant x$:

$$\int_a^{x} f(t)\mathrm{d}t = \int_a^{a'} f(t)\mathrm{d}t + \int_{a'}^{x} f(t)\mathrm{d}t$$

Then we take the limit (as $x \to +\infty$).

Of course, if we are in the case of a continuous function $f :]a, b] \to \mathbb{R}$ with $b' \in]a, b]$, then we have a similar result, and in case of convergence:

$$\int_a^{b} f(t)\mathrm{d}t = \int_a^{b'} f(t)\mathrm{d}t + \int_{b'}^{b} f(t)\mathrm{d}t$$

In this case the convergence of the integral does not depend on b, but only on the behavior of f in the neighborhood of a. □

5.1.4 Linearity

The following result is an immediate consequence of the linearity of the usual integrals and limits.

Proposition 5.1.5 (Linearity of the Integral) *Let f and g be two continuous functions on $[a, +\infty[$, and λ, μ two real numbers. If the integrals $\int_a^{+\infty} f(t)\mathrm{d}t$ and $\int_a^{+\infty} g(t)\mathrm{d}t$ converge, then $\int_a^{+\infty} (\lambda f(t) + \mu g(t))\mathrm{d}t$ converges and*

$$\int_a^{+\infty} (\lambda f(t) + \mu g(t))\mathrm{d}t = \lambda \int_a^{+\infty} f(t)\mathrm{d}t + \mu \int_a^{+\infty} g(t)\mathrm{d}t$$

The same relations hold for functions on an interval $]a, b]$, unbounded at a.

Remark 5.1.6 *The converse in linearity is false, it is possible to find two functions f, g such that $\int_a^{+\infty} f + g$ converges, without $\int_a^{+\infty} f$, ni $\int_a^{+\infty} g$ converging.*

5.1.5 Positivity

Proposition 5.1.7 (Positivity of the Integral) *Let $f, g : [a, +\infty[\to \mathbb{R}$ be continuous functions, having a convergent integral.*

$$\text{If } f \leqslant g \text{ then } \int_a^{+\infty} f(t)\mathrm{d}t \leqslant \int_a^{+\infty} g(t)\mathrm{d}t$$

In particular, the integral (convergent) of a positive function is positive:

$$\text{If } f \geqslant 0 \text{ then } \int_a^{+\infty} f(t)\mathrm{d}t \geqslant 0$$

Once again, the same relations hold for functions defined on an interval $]a, b]$, unbounded at a, taking care to have $a < b$.

Remark 5.1.8 *If we do not want to distinguish between the two types of improper integrals on an interval $[a, +\infty[$ (or $]-\infty, b]$) on the one hand and $]a, b]$ (or $[a, b[$) on the other hand, then it is convenient to add the two ends to the numerical line:*

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$$

Thus, the interval $I = [a, b[$ with $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$ denotes the infinite interval $[a, +\infty[$ (if $b = +\infty$) or the finite interval $[a, b[$ (if $b < +\infty$). Similarly for an interval $I' =]a, b]$ with $a = -\infty$ or $a \in \mathbb{R}$.

5.1.6 Cauchy's Criterion

We conclude with a characterization of convergence that is a bit more delicate (which can be skipped on a first reading).

Let us first recall Cauchy's criterion for limits.

Recall : Let $f : [a, +\infty[\to \mathbb{R}$. Then $\lim\limits_{x\to+\infty} f(x)$ exists and is finite if and only if

$$\forall \varepsilon > 0 \quad \exists M \geqslant a \quad (u, v \geqslant M \Longrightarrow |f(u) - f(v)| < \varepsilon).$$

Theorem 5.1.9 (Cauchy's Criterion) *Let $f : [a, +\infty[\to \mathbb{R}$ be a continuous function. The improper integral $\int_a^{+\infty} f(t)\mathrm{d}t$ converges if and only if*

$$\forall \varepsilon > 0 \quad \exists M \geqslant a \quad \left(u, v \geqslant M \Longrightarrow \left|\int_u^v f(t)\mathrm{d}t\right| < \varepsilon\right)$$

Proof: It suffices to apply the above recall to the function $F(x) = \int_a^x f(t)\mathrm{d}t$ and noting that $|F(u) - F(v)| = \left|\int_u^v f(t)\mathrm{d}t\right|$. □

5.1.7 Case of Two Singular Points

We can consider doubly improper integrals, that is, when both ends of the interval of definition are singular points. It is just a matter of reducing to two integrals each having only one singular point.

Definition 5.1.10 *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$. Let $f :]a, b[\to \mathbb{R}$ be a continuous function. We say that the integral $\int_a^b f(t)\mathrm{d}t$ converges if there exists $c \in]a, b[$ such that the two improper integrals $\int_a^c f(t)\mathrm{d}t$ and $\int_c^b f(t)\mathrm{d}t$ converge. The value of this doubly improper integral is then*

$$\int_a^c f(t)\mathrm{d}t + \int_c^b f(t)\mathrm{d}t$$

Chasles' relations imply that the nature and value of this doubly improper integral do not depend on the choice of c, with $a < c < b$.

Remark 5.1.11 *Attention! If one of the two integrals $\int_a^c f(t)\mathrm{d}t$ or $\int_c^b f(t)\mathrm{d}t$ diverges, then $\int_a^b f(t)\mathrm{d}t$ diverges. Take the example of $\int_{-x}^{+x} t\ \mathrm{d}t$ which always equals 0, yet $\int_{-\infty}^{+\infty} t\ \mathrm{d}t$ diverges! Indeed, whatever $c \in \mathbb{R}$, $\int_c^{+x} t\ \mathrm{d}t = \frac{x^2}{2} - \frac{c^2}{2}$ tends to $+\infty$ (as $x \to +\infty$).*

Example 5.1.12 *Does the following integral converge?*

$$\int_{-\infty}^{+\infty} \frac{t\ \mathrm{d}t}{(1+t^2)^2}$$

We choose (at random) $c = 2$. It is a matter of knowing whether the two integrals

$$\int_{-\infty}^{2} \frac{t\ \mathrm{d}t}{(1+t^2)^2} \quad \text{and} \quad \int_{2}^{+\infty} \frac{t\ \mathrm{d}t}{(1+t^2)^2}$$

converge.

Noting that a primitive of $\frac{t}{(1+t^2)^2}$ is $-\frac{1}{2}\frac{1}{1+t^2}$, we obtain:

$$\int_x^2 \frac{t\ \mathrm{d}t}{(1+t^2)^2} = -\frac{1}{2}\left[\frac{1}{1+t^2}\right]_x^2 = -\frac{1}{2}\left(\frac{1}{5} - \frac{1}{1+x^2}\right) \to -\frac{1}{10} \qquad \text{as } x \to -\infty.$$

Therefore $\int_{-\infty}^{2} \frac{t\ \mathrm{d}t}{(1+t^2)^2}$ converges and equals $-\frac{1}{10}$.

Similarly

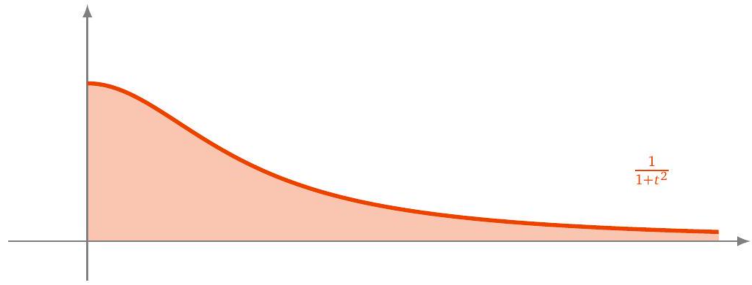
$$\int_2^x \frac{t\ \mathrm{d}t}{(1+t^2)^2} = -\frac{1}{2}\left[\frac{1}{1+t^2}\right]_2^x = -\frac{1}{2}\left(\frac{1}{1+x^2} - \frac{1}{5}\right) \to +\frac{1}{10} \qquad \text{as } x \to +\infty$$

Donc $\int_2^{+\infty} \frac{t\ \mathrm{d}t}{(1+t^2)^2}$ converge et vaut $+\frac{1}{10}$.

Thus $\int_{-\infty}^{+\infty} \frac{t\ \mathrm{d}t}{(1+t^2)^2}$ converges and equals $-\frac{1}{10} + \frac{1}{10} = 0$. This is not surprising since the function is an odd function. Repeat the calculations for another value of c and verify that the same result is obtained.

5.2 Positive Functions

We consider here $\int_a^{+\infty} f(t)\mathrm{d}t$, where f is of constant sign in the neighborhood of $+\infty$. Up to reducing the interval of integration, and possibly changing the sign of f if it is negative, we will assume that the function is positive or zero on the interval of integration $[a, +\infty[$.



Recall that, by definition,

$$\int_a^{+\infty} f(t)\mathrm{d}t = \lim_{x\to+\infty} \int_a^x f(t)\mathrm{d}t$$

Observe that if the function f is positive, then the primitive $\int_a^x f(t)\mathrm{d}t$ is an increasing function of x (because its derivative is $f(x)$). As x tends to infinity, either $\int_a^x f(t)\mathrm{d}t$ is bounded, and the integral $\int_a^{+\infty} f(t)\mathrm{d}t$ converges, or $\int_a^x f(t)\mathrm{d}t$ tends to $+\infty$.

5.2.1 Comparison Theorem

If we cannot (or do not want to) calculate a primitive of f, we study the convergence by comparing with integrals whose convergence is known, thanks to the following theorem.

Theorem 5.2.1 *Let f and g be two positive and continuous functions on $[a, +\infty[$. Assume that f is majorized by g in the neighborhood of $+\infty$:*

$$\exists A \geqslant a \quad \forall t > A \quad f(t) \leqslant g(t).$$

1. *If $\int_a^{+\infty} g(t)\mathrm{d}t$ converges then $\int_a^{+\infty} f(t)\mathrm{d}t$ converges.*
2. *If $\int_a^{+\infty} f(t)\mathrm{d}t$ diverges then $\int_a^{+\infty} g(t)\mathrm{d}t$ diverges.*

Proof: As we have observed, the convergence of integrals does not depend on the left bound of the interval, and we can simply study $\int_A^x f(t)\mathrm{d}t$ and $\int_A^x g(t)\mathrm{d}t$. Now using the positivity of the integral, we obtain that, for all $x \geqslant A$,

$$\int_A^x f(t)\mathrm{d}t \leqslant \int_A^x g(t)\mathrm{d}t$$

If $\int_A^{+\infty} g(t)\mathrm{d}t$ converges, then $\int_A^x f(t)\mathrm{d}t$ is an increasing function and majorized by $\int_A^{+\infty} g(t)\mathrm{d}t$, therefore convergent. Conversely, if $\int_A^x f(t)\mathrm{d}t$ tends to $+\infty$, then $\int_A^x g(t)\mathrm{d}t$ tends to $+\infty$ as well.
□

Here is a typical application of the comparison theorem 5.2.1.

Example 5.2.2 *Show that the integral*

$$\int_1^{+\infty} t^\alpha e^{-t}\ \mathrm{d}t \text{ converges,}$$

for any real α.

- *To do this, we first write:* $t^\alpha e^{-t} = t^\alpha e^{-t/2} e^{-t/2}$.
- *We know that* $\lim\limits_{t\to+\infty} t^\alpha e^{-t/2} = 0$, *for any* α, *because the exponential dominates the powers of* t. *Indeed, to compute the limit*

$$\lim_{t\to+\infty} t^\alpha e^{-t/2},$$

where $\alpha \in \mathbb{R}$, *we proceed as follows:*

1. *The term* t^α *grows without bound if* $\alpha > 0$, *tends to* 0 *if* $\alpha < 0$, *or remains constant if* $\alpha = 0$. *- The term* $e^{-t/2}$ *decays exponentially to* 0 *as* $t \to +\infty$.
2. *Resolving the competition The exponential decay of* $e^{-t/2}$ *dominates the polynomial growth of* t^α, *regardless of the value of* α. *Thus, we expect the limit to be* 0.
3. *Rigorous justification via substitution Let* $x = t/2$, *so as* $t \to +\infty$, *we have* $x \to +\infty$. *Rewriting the expression:*

$$t^\alpha e^{-t/2} = (2x)^\alpha e^{-x} = 2^\alpha x^\alpha e^{-x}.$$

We analyze the limit of $x^\alpha e^{-x}$ *as* $x \to +\infty$. *Using growth comparisons, we note that:*

$$x^\alpha \text{ is negligible compared to } e^x \text{ for any } \alpha \in \mathbb{R}.$$

Conclusion:
Therefore, the limit is:

$$\lim_{t\to+\infty} t^\alpha e^{-t/2} = 0, \quad \text{for any } \alpha \in \mathbb{R}.$$

- *In particular, there exists a real number* $A > 0$ *such that:*

$$\forall t > A \quad t^\alpha e^{-t/2} \leqslant 1.$$

- *Multiplying both sides of the inequality by* $e^{-t/2}$ *we obtain:*

$$\forall t > A \quad t^\alpha e^{-t} \leqslant e^{-t/2}.$$

- *Now the integral* $\displaystyle\int_1^{+\infty} e^{-t/2}\,\mathrm{d}t$ *converges. Indeed:*

$$\int_1^x e^{-t/2}\,\mathrm{d}t = \left[-2e^{-t/2}\right]_1^x = 2e^{-1/2} - 2e^{-x/2} \quad \text{and} \quad \lim_{x\to+\infty} 2e^{-1/2} - 2e^{-x/2} = 2e^{-1/2}$$

- *We can therefore apply the comparison theorem 5.2.1: since* $\displaystyle\int_1^{+\infty} e^{-t/2}\,\mathrm{d}t$ *converges, we deduce that* $\displaystyle\int_1^{+\infty} t^\alpha e^{-t}\,\mathrm{d}t$ *also converges.*

5.2.2 Equivalent Theorem

Thanks to the comparison theorem 5.2.1, we can replace the function to be integrated by an equivalent in the neighborhood of $+\infty$ to study the convergence of an integral.

Theorem 5.2.3 (Equivalent Theorem) *Let f and g be two continuous and strictly positive functions on $[a, +\infty[$. Assume that they are equivalent in the neighborhood of $+\infty$, that is:*

$$\lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1$$

Then the integral $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$ converges if and only if $\displaystyle\int_a^{+\infty} g(t)\mathrm{d}t$ converges.

Attention: it is important that f and g are positive!
We will denote the fact that f and g are equivalent in the neighborhood of $+\infty$ by: $f(t) \underset{+\infty}{\sim} g(t)$.

Proof: To say that two functions are equivalent in the neighborhood of $+\infty$ is to say that their ratio tends to 1, or again:

$$\forall \varepsilon > 0 \quad \exists A > a \quad \forall t > A \quad \left| \frac{f(t)}{g(t)} - 1 \right| < \varepsilon$$

or again:

$$\forall \varepsilon > 0 \quad \exists A > a \quad \forall t > A \quad (1 - \varepsilon)g(t) < f(t) < (1 + \varepsilon)g(t).$$

Fix $\varepsilon < 1$, and apply the comparison theorem 5.2.1 on the interval $[A, +\infty[$. If the integral $\displaystyle\int_A^{+\infty} f(t)\mathrm{d}t$ converges, then the integral $\displaystyle\int_A^{+\infty} (1-\varepsilon)g(t)\mathrm{d}t$ converges, so the integral $\displaystyle\int_A^{+\infty} g(t)\mathrm{d}t$ also converges by linearity.
Conversely, if $\displaystyle\int_A^{+\infty} f(t)\mathrm{d}t$ diverges, then $\displaystyle\int_A^{+\infty} (1+\varepsilon)g(t)\mathrm{d}t$ diverges, so $\displaystyle\int_A^{+\infty} g(t)\mathrm{d}t$ also diverges.
□

Example 5.2.4 *Does the integral*

$$\int_1^{+\infty} \frac{t^5 + 3t + 1}{t^3 + 4} e^{-t} \, \mathrm{d}t \qquad \textit{converge ?}$$

As

$$\frac{t^5 + 3t + 1}{t^3 + 4} e^{-t} \underset{+\infty}{\sim} t^2 e^{-t}$$

and we have already shown that the integral $\displaystyle\int_1^{+\infty} t^2 e^{-t} \, \mathrm{d}t$ converges, then our integral converges.

5.2.3 Riemann Integrals

For the study of the convergence of an integral for which we do not have a primitive, the use of equivalents allows us to reduce to a catalog of integrals whose nature is known. The most classic are the Riemann and Bertrand integrals.
A Riemann integral is:

$$\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t$$

where $\alpha \in \mathbb{R}$.
In this case, the primitive is explicit:

$$\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t = \begin{cases} \lim\limits_{x \to +\infty} \left[\dfrac{1}{-\alpha+1} \dfrac{1}{t^{\alpha-1}} \right]_1^x & \text{if } \alpha \neq 1 \\ \lim\limits_{x \to +\infty} [\ln t]_1^x & \text{if } \alpha = 1 \end{cases}$$

We immediately deduce the nature (convergent or divergent) of the Riemann integrals.

$$\begin{array}{ll} \text{If} \quad \alpha > 1 \text{ then} \displaystyle\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t & \text{converges.} \\ \text{If} \quad \alpha \leqslant 1 \text{ then} \displaystyle\int_1^{+\infty} \frac{1}{t^\alpha} \mathrm{d}t & \text{diverges.} \end{array}$$

5.2.4 Bertrand Integrals

A Bertrand integral is

$$\int_2^{+\infty} \frac{1}{t(\ln t)^\beta} \mathrm{d}t,$$

where $\beta \in \mathbb{R}$.
The primitive is explicit:

$$\int_2^{+\infty} \frac{1}{t(\ln t)^\beta} \mathrm{d}t = \begin{cases} \lim\limits_{x \to +\infty} \left[\dfrac{1}{-\beta+1} (\ln t)^{-\beta+1} \right]_2^x & \text{if} \quad \beta \neq 1 \\ \lim\limits_{x \to +\infty} [\ln(\ln t)]_2^x & \text{if} \quad \beta = 1 \end{cases}$$

We deduce the nature of the Bertrand integrals.

$$\begin{array}{ll} \text{If } \beta > 1 \text{ then} & \displaystyle\int_2^{+\infty} \frac{1}{t(\ln t)^\beta} \mathrm{d}t \text{ converges.} \\ \text{If } \beta \leqslant 1 \text{ then} & \displaystyle\int_2^{+\infty} \frac{1}{t(\ln t)^\beta} \mathrm{d}t \text{ diverges.} \end{array}$$

Here is an example of an application:

Example 5.2.5 *Does the integral*

$$\int_2^{+\infty} \sqrt{t^2+3t} \ln\left(\cos\frac{1}{t}\right) \sin^2\left(\frac{1}{\ln t}\right) \mathrm{d}t \qquad \textit{converge ?}$$

The singular point is $+\infty$. To answer the question, let's calculate an equivalent of the function in the neighborhood of $+\infty$. We have:

$$\sqrt{t^2+3t} = t\sqrt{1+\frac{3}{t}} \underset{+\infty}{\sim} t$$

$$\ln\left(\cos\frac{1}{t}\right) = \ln\left(1 - \frac{1}{2t^2} + o\left(\frac{1}{t^2}\right)\right) \underset{+\infty}{\sim} -\frac{1}{2t^2}$$

$$\sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} \left(\frac{1}{\ln t}\right)^2$$

Hence an equivalent of the function in the neighborhood of $+\infty$:

$$\sqrt{t^2+3t} \ln\left(\cos\frac{1}{t}\right) \sin^2\left(\frac{1}{\ln t}\right) \underset{+\infty}{\sim} -\frac{1}{2t(\ln t)^2}$$

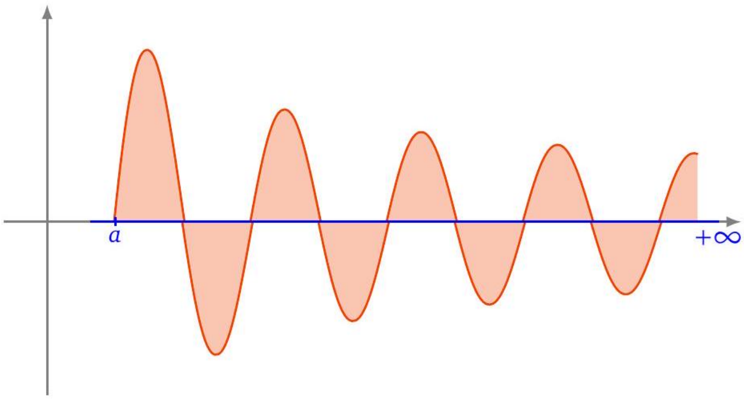
Let us note that in this equivalence, both functions are negative in the neighborhood of $+\infty$. According to Theorem 5.2.3, the two associated integrals have the same nature. Since Bertrand's integral

$$\int_2^{+\infty} \frac{1}{t(\ln t)^2}\,\mathrm{d}t$$

converges, our initial integral is therefore also convergent.

5.3 Oscillating Functions

We consider here $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$, where $f(t)$ oscillates to infinity between positive and negative values.



The definition of the improper integral remains the same:

$$\int_a^{+\infty} f(t)\mathrm{d}t = \lim_{x\to+\infty} \int_a^x f(t)\mathrm{d}t$$

Unlike the case of positive functions, where the limit was either finite or equal to $+\infty$, all behaviors are possible here: the values of $\displaystyle\int_a^x f(t)\mathrm{d}t$ can tend towards a finite limit, towards $+\infty$ or $-\infty$, or even oscillate between two finite values (like $\displaystyle\int_a^x \sin t\ \mathrm{d}t$), or approach alternately $+\infty$ and $-\infty$ (like $\displaystyle\int_a^x t\sin t\ \mathrm{d}t$).

5.3.1 Absolutely Convergent Integral

The most favorable case is when the absolute value of f converges.

Definition 5.3.1 *Let f be a continuous function on $[a, +\infty[$. We say that $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$ is absolutely convergent if $\displaystyle\int_a^{+\infty} |f(t)|\mathrm{d}t$ converges.*

The following theorem is often used to prove the convergence of an integral. Unfortunately, it does not allow us to calculate the value of this integral.

Theorem 5.3.2 *If the integral $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$ is absolutely convergent, then it is convergent.*

In other words, being absolutely convergent is stronger than being convergent.

Proof: This is a consequence of Cauchy's criterion (theorem 5.1.9) applied to $|f|$, then to f. As $\displaystyle\int_a^{+\infty} |f(t)|\mathrm{d}t$ converges, then by Cauchy's criterion (direct sense):

$$\forall\varepsilon > 0 \quad \exists M \geqslant a \quad \left(u, v \geqslant M \Longrightarrow \int_u^v |f(t)|\mathrm{d}t < \varepsilon\right)$$

But since

$$\left|\int_u^v f(t)\mathrm{d}t\right| \leqslant \int_u^v |f(t)|\mathrm{d}t < \varepsilon$$

then by Cauchy's criterion (converse sense), $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$ converges. □

Example 5.3.3 *For example,*

$$\int_1^{+\infty} \frac{\sin t}{t^2}\,\mathrm{d}t \qquad \textit{is absolutely convergent,}$$

hence convergent. Indeed, for all t,

$$\frac{|\sin t|}{t^2} \leqslant \frac{1}{t^2}.$$

Now the Riemann integral $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ *is convergent. Hence the result by the comparison theorem 5.2.1.*

5.3.2 Semi-convergent Integral

Definition 5.3.4 *An integral* $\displaystyle\int_a^{+\infty} f(t)\mathrm{d}t$ *is semi-convergent if it is convergent but not absolutely convergent.*

Example 5.3.5 *We will prove that it is convergent, but not absolutely convergent.*

$$\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t \qquad \textit{is semi-convergent.}$$

1. ***The integral is convergent.***
   To show this, let's perform integration by parts (with $u' = \sin t, v = \frac{1}{t}$*):*

$$\int_1^x \frac{\sin t}{t}\,\mathrm{d}t = \left[\frac{-\cos t}{t}\right]_1^x - \int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t$$

   Let's examine the two terms:

   - $\left[\frac{-\cos t}{t}\right]_1^x = -\frac{\cos x}{x} + \cos 1$. *Now the function* $\frac{\cos x}{x}$ *tends to 0 (as* $x \to +\infty$*), because* $\cos x$ *is bounded and* $\frac{1}{x}$ *tends to 0. Therefore* $\left[\frac{-\cos t}{t}\right]_1^x$ *has a finite limit (which is* $\cos 1$*).*
   - *For the other term, let's first note that* $\displaystyle\int_1^{+\infty} \frac{\cos t}{t^2}\,\mathrm{d}t$ *is an absolutely convergent integral. Indeed* $\frac{|\cos t|}{t^2} \leqslant \frac{1}{t^2}$ *and the Riemann integral* $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ *converges.*

Therefore, $\int_1^{+\infty} \frac{\cos t}{t^2}\,\mathrm{d}t$ *converges, which means exactly that* $\int_1^x \frac{\cos t}{t^2}\,\mathrm{d}t$ *has a finite limit.*
Conclusion: $\int_1^x \frac{\sin t}{t}\,\mathrm{d}t$ *has a finite limit (as* $x \to +\infty$ *), and therefore by definition* $\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ *converges.*

2. ***The integral is not absolutely convergent.***
Here's a way to check it. As $|\sin t| \leqslant 1$ *for all* t*, we have:*

$$\frac{|\sin t|}{t} \geqslant \frac{\sin^2 t}{t} = \frac{1-\cos(2t)}{2t}$$

Applying integration by parts to $\frac{\cos(2t)}{t}$ $\left(\text{avec}\, u' = \cos(2t) \text{ and } v = \frac{1}{t}\right)$*, we obtain:*

$$\int_1^x \frac{1-\cos(2t)}{2t}\,\mathrm{d}t = \frac{1}{2}[\ln t]_1^x - \frac{1}{4}\left[\frac{\sin(2t)}{t}\right]_1^x - \frac{1}{4}\int_1^x \frac{\sin(2t)}{t^2}\,\mathrm{d}t$$

Now $\int_1^{+\infty} \frac{\sin(2t)}{t^2}\,\mathrm{d}t$ *converges absolutely. Of the three terms in the sum above, the last two converge, and the first tends to* $+\infty$*. Therefore the integral diverges, and by the comparison theorem 5.2.1, the integral* $\int_1^{+\infty} \frac{|\sin t|}{t}\,\mathrm{d}t$ *also diverges.*

5.3.3 Abel's Theorem

To show that an integral converges, when it is not absolutely convergent, we have the following theorem.

Theorem 5.3.6 (Abel's Theorem) *Let* f *be a* $\mathscr{C}^1$ *function on* $[a,+\infty[$*, positive, decreasing, having a limit of zero at* $+\infty$*. Let* g *be a continuous function on* $[a,+\infty[$ *, such that the primitive* $\int_a^x g(t)\mathrm{d}t$ *is bounded. Then the integral*

$$\int_a^{+\infty} f(t)g(t)\mathrm{d}t \quad \textit{converges.}$$

Remark 5.3.7 *With* $f(t) = \frac{1}{t}$ *and* $g(t) = \sin t$*, we find that the integral* $\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ *converges.*

Proof: This is a generalization of the previous example 5.3.5. For all $x \geqslant a$, let $G(x) = \int_a^x g(t)\mathrm{d}t$. By hypothesis, G is bounded, so there exists M such that, for all $x, |G(x)| \leqslant M$. Now let's perform integration by parts:

$$\int_a^x f(t)g(t)\mathrm{d}t = [f(t)G(t)]_a^x - \int_a^x f'(t)G(t)\mathrm{d}t.$$

As G is bounded and f tends to 0, the bracketed term converges. Now let's show that the second term also converges, by verifying that

$$\int_a^{+\infty} f'(t)G(t)\mathrm{d}t \quad \text{is absolutely convergent.}$$

We have:

$$\left|f'(t)G(t)\right| = \left|f'(t)\right| |G(t)| \leqslant \left(-f'(t)\right) M,$$

because f is decreasing (so $f'(t) \leqslant 0$) and $|G|$ is bounded by M. By the comparison theorem 5.2.1, it is therefore sufficient to show that $\int_a^{+\infty} (-f'(t))\mathrm{d}t$ is convergent.

Now:

$$\int_a^x (-f'(t))\mathrm{d}t = f(a) - f(x) \quad \text{and} \quad \lim_{x\to+\infty} (f(a) - f(x)) = f(a)$$

□

Example 5.3.8 *As an example of an application, if α is a strictly positive real number, and k a positive odd integer, then the integral*

$$\int_1^{+\infty} \frac{\sin^k(t)}{t^\alpha}\mathrm{d}t \text{ converges.}$$

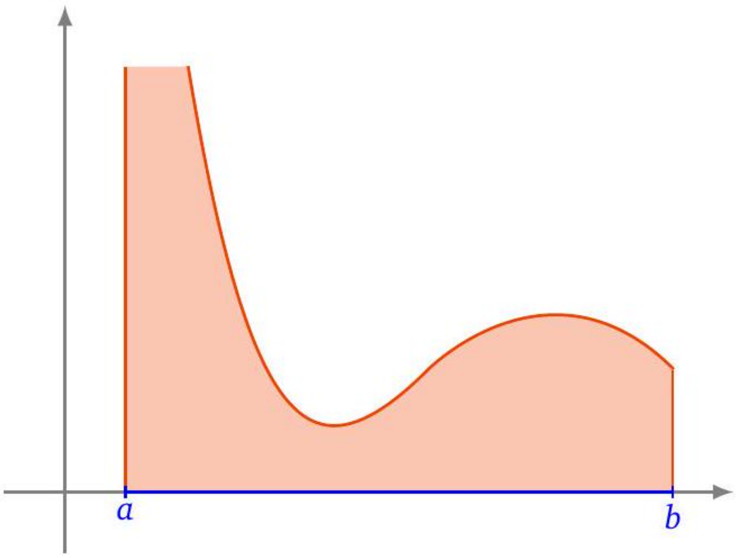
Notice that this integral is absolutely convergent only for $\alpha > 1$. We verify that the hypotheses of theorem 5.3.6 are satisfied for $f(t) = \frac{1}{t^\alpha}$ and $g(t) = \sin^k(t)$. To ensure that the primitive of $\sin^k$ is bounded, it suffices to think of a linearization, which will transform $\sin^k(t)$ into a linear combination of the $\sin(\ell t), \ell = 1, \dots, k$, whose primitive will always be bounded.

5.4 Improper Integrals on a Bounded Interval

5.4.1 Positive Functions

We deal here with the case where the function to be integrated tends to infinity at one of the bounds of the interval of integration. The treatment is quite analogous to the case of a positive function on an unbounded interval, and we will omit the proofs.

Up to reducing the interval of integration, and possibly changing the sign of f, we can assume that the function is positive or zero on the interval of integration $]a, b]$, and tends to $+\infty$ at a.



Recall that, by definition,

$$\int_a^b f(t)\mathrm{d}t = \lim_{x\to a^+} \int_x^b f(t)\mathrm{d}t$$

Observe that if the function f is positive, then $\int_x^b f(t)\mathrm{d}t$ increases as x decreases towards a: either $\int_x^b f(t)\mathrm{d}t$ is bounded, and the integral $\int_a^b f(t)\mathrm{d}t$ is convergent, or $\int_x^b f(t)\mathrm{d}t$ tends to $+\infty$.

5.4.2 Comparison Theorem

Theorem 5.4.1 *Let f and g be two positive and continuous functions on $]a,b]$. Assume that f is majorized by g in the neighborhood of a, that is:*

$$\exists \varepsilon > 0 \quad \forall t \in]a, a+\varepsilon] \quad f(t) \leqslant g(t).$$

1. *If $\displaystyle\int_a^b g(t)\mathrm{d}t$ converges then $\displaystyle\int_a^b f(t)\mathrm{d}t$ converges.*

2. *If $\displaystyle\int_a^b f(t)\mathrm{d}t$ diverges then $\displaystyle\int_a^b g(t)\mathrm{d}t$ diverges.*

Proof: Let $\varepsilon > 0$ be as in the hypothesis. The integrals are improper at $x = a$ only (the functions are continuous on $]a,b]$, and b is a regular endpoint).

Define for $x \in (a,b]$:

$$F(x) = \int_x^b f(t)\,dt, \qquad G(x) = \int_x^b g(t)\,dt.$$

By definition, $\displaystyle\int_a^b f(t)\,dt$ converges iff $\displaystyle\lim_{x\to a^+} F(x)$ exists and is finite, and similarly for g and G.

Monotonicity and inequality near a. Since $f, g \geq 0$, F and G are decreasing functions of x (as x decreases to a, the domain of integration increases). Take $x \in (a, a+\varepsilon]$. Then

$$F(x) = \int_x^{a+\varepsilon} f(t)\,dt \;+\; \int_{a+\varepsilon}^b f(t)\,dt,$$

$$G(x) = \int_x^{a+\varepsilon} g(t)\,dt \;+\; \int_{a+\varepsilon}^b g(t)\,dt.$$

The term $\displaystyle\int_{a+\varepsilon}^b f(t)\,dt$ is an ordinary Riemann integral (continuous on a closed interval), hence finite; denote it by C_f. Similarly, denote $C_g = \displaystyle\int_{a+\varepsilon}^b g(t)\,dt$.

For the integrals from x to $a+\varepsilon$, since on $[x, a+\varepsilon]$ we have $f(t) \leq g(t)$,

$$\int_x^{a+\varepsilon} f(t)\,dt \leq \int_x^{a+\varepsilon} g(t)\,dt.$$

Thus

$$F(x) \leq \int_x^{a+\varepsilon} g(t)\,dt + C_f, \qquad G(x) = \int_x^{a+\varepsilon} g(t)\,dt + C_g.$$

Proof of part (1). Assume $\int_a^b g(t)\,dt$ converges, i.e. $\displaystyle\lim_{x\to a^+} G(x) = L_g$ finite. Then $\displaystyle\lim_{x\to a^+} \int_x^{a+\varepsilon} g(t)\,dt = L_g - C_g$ is finite. Since $F(x) \leq \Big(\displaystyle\int_x^{a+\varepsilon} g(t)\,dt\Big) + C_f$, and the right-hand side has a finite limit as $x \to a^+$, and $F(x)$ is decreasing and bounded below (by 0), the limit $\displaystyle\lim_{x\to a^+} F(x)$ exists and is finite. Hence $\displaystyle\int_a^b f(t)\,dt$ converges.

Proof of part (2). Assume $\displaystyle\int_a^b f(t)\,dt$ diverges. Since $f \geq 0$ and continuous, divergence means

$$\lim_{x\to a^+} F(x) = +\infty.$$

From the inequality $f(t) \leq g(t)$ on $]a, a+\varepsilon]$, we have for $x \in (a, a+\varepsilon]$:

$$\int_x^{a+\varepsilon} f(t)\,dt \leq \int_x^{a+\varepsilon} g(t)\,dt.$$

Then

$$F(x) = \int_x^{a+\varepsilon} f(t)\,dt + C_f \quad \Longrightarrow \quad \int_x^{a+\varepsilon} f(t)\,dt = F(x) - C_f.$$

Thus

$$F(x) - C_f \leq \int_x^{a+\varepsilon} g(t)\,dt.$$

As $x \to a^+$, $F(x) \to +\infty$, so the left-hand side tends to $+\infty$; hence $\int_x^{a+\varepsilon} g(t)\,dt \to +\infty$ as $x \to a^+$. Therefore $G(x) = \int_x^{a+\varepsilon} g(t)\,dt + C_g \to +\infty$, so $\int_a^b g(t)\,dt$ diverges.

Conclusion. We have shown both implications, completing the proof. □

Example 5.4.2 *Fix a real number α. Does the integral*

$$\int_0^1 \frac{(-\ln t)^\alpha}{\sqrt{t}}\,\mathrm{d}t \qquad \textit{converge?}$$

- *To find out, we write:*

$$\frac{(-\ln t)^\alpha}{\sqrt{t}} = \left((-\ln t)^\alpha t^{1/4}\right) t^{-3/4}$$

- *We know that $\lim_{t \to 0^+} (-\ln t)^\alpha t^{1/4} = 0$, for any α (the powers of t dominate the logarithm).*
- *In particular, there exists a real number $\varepsilon > 0$ such that:*

$$\forall t \in]0, \varepsilon] \quad (-\ln t)^\alpha t^{1/4} \leqslant 1.$$

- *Multiplying both sides of the inequality by $t^{-3/4}$ we obtain:*

$$\forall t \in]0, \varepsilon] \quad \frac{(-\ln t)^\alpha}{\sqrt{t}} \leqslant t^{-3/4}$$

- *Now the integral $\int_0^1 t^{-3/4}\,\mathrm{d}t$ converges. Indeed:*

$$\int_x^1 t^{-3/4}\,\mathrm{d}t = \left[4t^{1/4}\right]_x^1 = 4 - 4x^{1/4} \quad \textit{and} \quad \lim_{x \to 0^+}\left(4 - 4x^{1/4}\right) = 4$$

- *We can therefore apply the comparison theorem 5.4.1: since $\int_0^1 t^{-3/4}\,\mathrm{d}t$ converges, then $\int_0^1 \frac{(-\ln t)^\alpha}{\sqrt{t}}\,\mathrm{d}t$ also converges, whatever α.*

5.4.3 Equivalent Theorem

Thanks to the comparison theorem 5.4.1, we can replace the function to be integrated by an equivalent in the neighborhood of a to study the convergence of an integral.

Theorem 5.4.3 *Let f and g be two continuous and strictly positive functions on $]a, b]$. Assume that they are equivalent in the neighborhood of a, that is:*

$$\lim_{t \to a^+} \frac{f(t)}{g(t)} = 1$$

Then the integral $\int_a^b f(t)\mathrm{d}t$ converges if and only if $\int_a^b g(t)\mathrm{d}t$ converges.

Proof: Since $\lim_{t \to a^+} \frac{f(t)}{g(t)} = 1$, by definition of the limit, for $\varepsilon = \frac{1}{2}$ there exists $\delta > 0$ such that for all $t \in (a, a + \delta]$ we have

$$\left| \frac{f(t)}{g(t)} - 1 \right| < \frac{1}{2}.$$

This implies

$$\frac{1}{2} < \frac{f(t)}{g(t)} < \frac{3}{2} \quad \text{for all } t \in (a, a + \delta].$$

Multiplying through by $g(t) > 0$ yields

$$\frac{1}{2} g(t) < f(t) < \frac{3}{2} g(t) \quad \text{for all } t \in (a, a + \delta].$$

Convergence of $\int_a^b g$ implies convergence of $\int_a^b f$. Assume $\int_a^b g(t)\, dt$ converges. Since $f(t) < \frac{3}{2} g(t)$ on $(a, a+\delta]$, by the comparison theorem (Theorem 5.4.1 with the majorant $\frac{3}{2} g$) the integral $\int_a^b f(t)\, dt$ also converges.

Convergence of $\int_a^b f$ implies convergence of $\int_a^b g$. Assume $\int_a^b f(t)\, dt$ converges. From $\frac{1}{2} g(t) < f(t)$ we obtain $g(t) < 2f(t)$ on $(a, a + \delta]$. Again by the comparison theorem (now with the majorant $2f$), the integral $\int_a^b g(t)\, dt$ converges.

Conclusion. We have shown both implications, hence the integrals converge or diverge together. □

Remark 5.4.4 *Attention: it is important that f and g are positive.*
The equivalence of f and g in the neighborhood of a will be denoted by: $f(t) \underset{a}{\sim} g(t)$ (or $f(t) \underset{a^+}{\sim} g(t)$ to specify that the limit at a is the right-hand limit).

Example 5.4.5 *The integral*

$$\int_0^1 \sqrt{\frac{-\ln t + 1}{\sin t}}\, \mathrm{d}t \qquad \textit{converges.}$$

Indeed,

$$\sqrt{\frac{-\ln t + 1}{\sin t}} \underset{0^+}{\sim} \frac{(-\ln t)^{1/2}}{\sqrt{t}},$$

and we have already shown that the integral $\int_0^1 \frac{(-\ln t)^{1/2}}{\sqrt{t}}\, \mathrm{d}t$ converges.

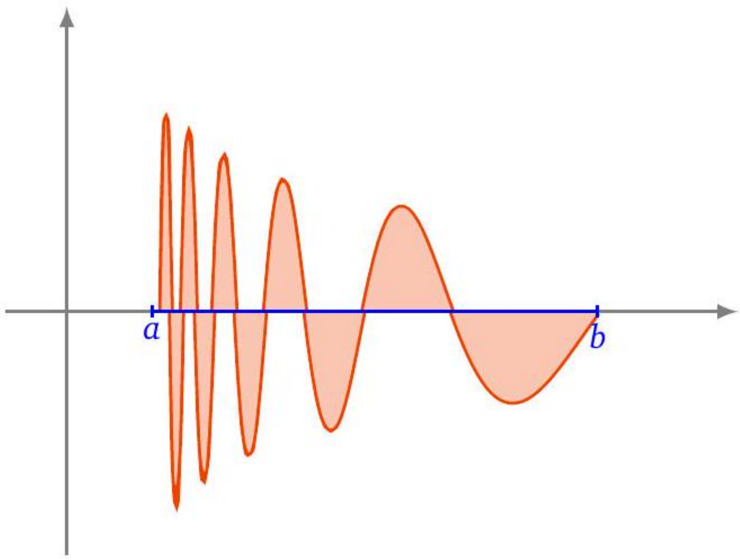
The use of equivalents thus allows us to reduce the study of the convergence of an integral for which we do not have a primitive to a catalog of integrals whose convergence is known. The most classic are of the type $\int_0^1 \frac{1}{t^\alpha}\mathrm{d}t$, but be careful, the convergence as a function of the parameter α is reversed compared to the Riemann integrals.

Example 5.4.6

$$\text{If } \alpha < 1 \quad \text{then} \quad \int_0^1 \frac{1}{t^\alpha}\mathrm{d}t \quad \text{converges.}$$
$$\text{If } \alpha \geqslant 1 \quad \text{then} \quad \int_0^1 \frac{1}{t^\alpha}\mathrm{d}t \quad \text{diverges.}$$

5.4.4 Oscillating Functions

The last case to be treated is that where the function to be integrated oscillates in the neighborhood of one of the bounds, taking values arbitrarily close to $+\infty$ or $-\infty$.



The change of variable $u = \dfrac{1}{t-a}$ allows us to reduce to the previous case of an oscillating function on an unbounded interval, which will dispense us from giving as much detail.

Recall that, by definition,

$$\int_a^b f(t)\mathrm{d}t = \lim_{x\to a^+} \int_x^b f(t)\mathrm{d}t$$

The important notion is still absolute convergence.

Definition 5.4.7 *Let f be a continuous function on $]a,b]$. We say that $\int_a^b f(t)\mathrm{d}t$ is absolutely convergent if $\int_a^b |f(t)|\mathrm{d}t$ is a convergent integral.*

The following theorem is proved in the same way as theorem 5.3.2.

Theorem 5.4.8 *If the integral $\int_a^b f(t)\mathrm{d}t$ is absolutely convergent, then it is convergent.*

Example 5.4.9

1. *The integral*

$$\int_0^1 \frac{\sin \frac{1}{t}}{\sqrt{t}}\,\mathrm{d}t \quad \text{is absolutely convergent,}$$

hence convergent. Indeed, for all t,

$$\frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}} \leqslant \frac{1}{\sqrt{t}}$$

Now the integral $\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t$ *converges, hence the result by the comparison theorem 5.4.1.*

2. *On the other hand,*

$$\int_0^1 \frac{\sin \frac{1}{t}}{t}\,\mathrm{d}t \quad \text{is not absolutely convergent,}$$

but it is convergent. To see this, let's perform the change of variable $t \mapsto \frac{1}{u}$*:*

$$\int_x^1 \frac{\sin \frac{1}{t}}{t}\,\mathrm{d}t = \int_{1/x}^1 u \sin u \frac{-1}{u^2}\,\mathrm{d}u = \int_1^{1/x} \frac{\sin u}{u}\,\mathrm{d}u$$

As $x \to 0^+$ *then* $\frac{1}{x} \to +\infty$. *We have already shown that the integral* $\int_1^{+\infty} \frac{\sin u}{u}\,\mathrm{d}u$ *is convergent, without being absolutely convergent.*

We could state an Abel's theorem analogous to theorem 5.3.6, but this is not really useful. On the one hand, the functions to which it would apply are rarely encountered, and on the other hand, it is generally easy to reduce to a problem on $[c, +\infty\,[$, *by the change of variable* $t \mapsto u = \frac{1}{t-a}$: *we have already done this for* $\int_0^1 \frac{\sin \frac{1}{t}}{t}\,\mathrm{d}t$.

5.5 Integration by Parts - Change of Variable

5.5.1 Integration by Parts

Theorem 5.5.1 *Let* u *and* v *be two functions of class* $\mathscr{C}^1$ *on the interval* $[a, +\infty\,[$. *Assume that* $\lim\limits_{t \to +\infty} u(t)v(t)$ *exists and is finite. Then the integrals* $\int_a^{+\infty} u(t)v'(t)\mathrm{d}t$ *and* $\int_a^{+\infty} u'(t)v(t)\mathrm{d}t$ *are of the same nature. In case of convergence we a :*

$$\int_a^{+\infty} u(t)v'(t)\mathrm{d}t = [uv]_a^{+\infty} - \int_a^{+\infty} u'(t)v(t)\mathrm{d}t$$

Recall that $[uv]_a^{+\infty} = \lim\limits_{t \to +\infty} (uv)(t) - (uv)(a)$.

The best approach is not to apply the theorem directly, but to perform the proof each time, i.e., by doing integration by parts on the interval $[a, x]$ and verifying that the objects have a limit as $x \to +\infty$.

Proof: This is the usual integration by parts formula

$$\int_a^x u(t)v'(t)\mathrm{d}t = [uv]_a^x - \int_a^x u'(t)v(t)\mathrm{d}t,$$

noting that by assumption the bracket has a finite limit as $x \to +\infty$. □

Example 5.5.2 *Let $\lambda > 0$. What is the expectation of the exponential distribution:*

$$\int_0^{+\infty} \lambda t e^{-\lambda t} \ \mathrm{d}t$$

We perform integration by parts with $u = \lambda t, v' = e^{-\lambda t}$. We have therefore $u' = \lambda$ and $v = \frac{-1}{\lambda} e^{-\lambda t}$. Thus

$$\begin{aligned}
\int_0^x \lambda t e^{-\lambda t} \ \mathrm{d}t &= \int_0^x u(t)v'(t)\mathrm{d}t \\
&= [uv]_0^x - \int_0^x u'(t)v(t)\mathrm{d}t \\
&= \left[\lambda t \cdot \frac{-1}{\lambda} e^{-\lambda t}\right]_0^x - \int_0^x \lambda \cdot \frac{-1}{\lambda} e^{-\lambda t} \ \mathrm{d}t \\
&= -x e^{-\lambda x} + \int_0^x e^{-\lambda t}\mathrm{d}t \\
&= -x e^{-\lambda x} + \left[\frac{-1}{\lambda} e^{-\lambda t}\right]_0^x \\
&= -x e^{-\lambda x} - \frac{1}{\lambda}\left(e^{-\lambda x} - 1\right) \\
&\longrightarrow \frac{1}{\lambda} \quad \text{when } x \to +\infty
\end{aligned}$$

Thus the integral converges and

$$\int_0^{+\infty} \lambda t e^{-\lambda t} \ \mathrm{d}t = \frac{1}{\lambda}$$

5.5.2 Change of Variable

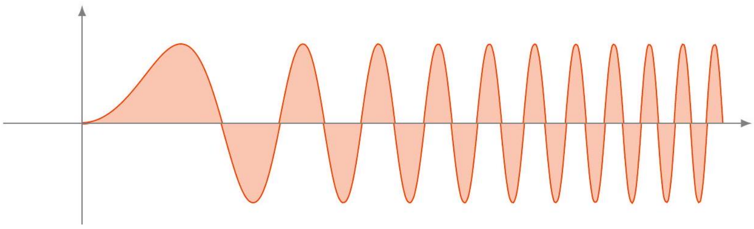
Theorem 5.5.3 *Let f be a function defined on an interval $I = [a, +\infty[$. Let $J = [\alpha, \beta[$ be an interval with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ or $\beta = +\infty$. Let $\varphi : J \to I$ be a diffeomorphism of class $\mathscr{C}^1$. The integrals $\int_a^{+\infty} f(x)\mathrm{d}x$ and $\int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\mathrm{d}t$ are of the same nature. In case of convergence, we have:*

$$\int_a^{+\infty} f(x)\mathrm{d}x = \int_\alpha^\beta f(\varphi(t)) \cdot \varphi'(t)\mathrm{d}t$$

The proof is the same as for the usual change of variable. Again, it's better not to apply the theorem directly, but to perform a classical change of variable on the interval $[a, x]$, then study the limits as $x \to +\infty$.
Recall that $\varphi : J \to I$ is a diffeomorphism of class $\mathscr{C}^1$ if φ is a $\mathscr{C}^1$ application, bijective, whose reciprocal bijection is also $\mathscr{C}^1$.
The following example is particularly interesting: the function $f(t) = \sin\left(t^2\right)$ has a convergent integral, but does not tend to 0 (as $t \to +\infty$). This is in contrast to the case of series: for a convergent series, the general term always tends to 0.

Example 5.5.4 *The Fresnel integral*

$$\int_1^{+\infty} \sin\left(t^2\right) \mathrm{d}t \ \textit{converges.}$$



We perform the change of variable $u = t^2$*, which implies* $t = \sqrt{u}$*,* $\mathrm{d}t = \dfrac{\mathrm{d}u}{2\sqrt{u}} \cdot \varphi : u \mapsto t = \sqrt{u}$ *is a diffeomorphism between* $u \in [1, x^2]$ *and* $t \in [1, x]$*. Hence*

$$\int_1^x \sin(t^2)\,\mathrm{d}t = \int_1^{x^2} \sin(u)\frac{\mathrm{d}u}{2\sqrt{u}}$$

Now, by Abel's theorem $\displaystyle\int_1^{+\infty} \frac{\sin u}{\sqrt{u}}\,\mathrm{d}u$ *converges, therefore* $\displaystyle\int_1^{x^2} \sin(u)\frac{\mathrm{d}u}{2\sqrt{u}}$ *admits a finite limit (as* $x \to +\infty$*), which proves that* $\displaystyle\int_1^x \sin(t^2)\,\mathrm{d}t$ *also admits a finite limit. Conclusion:* $\displaystyle\int_1^{+\infty} \sin(t^2)\,\mathrm{d}t$ *converges.*

Example 5.5.5 *Let's calculate the value of the two integrals*

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin t)\mathrm{d}t \quad J = \int_0^{\frac{\pi}{2}} \ln(\cos t)\mathrm{d}t$$

1. ***The integral*** I ***converges.*** *The uncertain point is at* $t = 0$*. Since* $\sin t \underset{0^+}{\sim} t, \ln t \leqslant \dfrac{1}{\sqrt{t}}$ *(for* t *small enough), and the integral* $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{t}}\,\mathrm{d}t$ *converges, then the integral* $\displaystyle\int_0^{\frac{\pi}{2}} \ln t\,\mathrm{d}t$ *converges, which implies that* I *converges.*

2. ***Let's check that*** $I = J$*. Let's make the change of variable* $t = \dfrac{\pi}{2} - u$*. We have* $\mathrm{d}t = -\mathrm{d}u$ *and a diffeomorphism between* $t \in \left[x, \dfrac{\pi}{2}\right]$ *and* $u \in \left[\dfrac{\pi}{2} - x, 0\right]$*. Thus*

$$\int_x^{\frac{\pi}{2}} \ln(\sin t)\mathrm{d}t = \int_{\frac{\pi}{2}-x}^{0} \ln\left(\sin\left(\frac{\pi}{2} - u\right)\right)(-\mathrm{d}u) = \int_0^{\frac{\pi}{2}-x} \ln(\cos u)\mathrm{d}u.$$

   Thus, as $x \to 0$*, this proves* $I = J$ *(and in particular* J *converges).*

3. ***Calculation of*** $I + J$***.***

$$\begin{aligned} I + J &= \int_0^{\frac{\pi}{2}} \ln(\sin t)\mathrm{d}t + \int_0^{\frac{\pi}{2}} \ln(\cos t)\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} (\ln(\sin t) + \ln(\cos t))\mathrm{d}t = \int_0^{\frac{\pi}{2}} \ln(\sin t \cdot \cos t)\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2t)\right)\mathrm{d}t \\ &= -\frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\sin(2t))\mathrm{d}t \end{aligned}$$

   And since $I = J$*, we have*

$$2I = -\frac{\pi}{2}\ln 2 + K.$$

We still need to evaluate $K = \int_0^{\frac{\pi}{2}} \ln(\sin(2t))\mathrm{d}t$*:*

$$\begin{aligned}
K &= \int_0^{\frac{\pi}{2}} \ln(\sin(2t))\mathrm{d}t \\
&= \frac{1}{2}\int_0^{\pi} \ln(\sin u)\mathrm{d}u \quad (\textit{ change of variable } u = 2t) \\
&= \frac{1}{2}I + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin u)\mathrm{d}u \\
&= \frac{1}{2}I + \frac{1}{2}\int_{\frac{\pi}{2}}^{0} \ln(\sin(\pi - v))(-\mathrm{d}v) \quad (\textit{ change of variable } v = \pi - u) \\
&= \frac{1}{2}I + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin v)\mathrm{d}v \\
&= \frac{1}{2}I + \frac{1}{2}I = I.
\end{aligned}$$

4. ***Conclusion.*** *Thus, since* $2I = -\frac{\pi}{2}\ln 2 + K$ *and* $K = I$ *we find:*

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin t)\mathrm{d}t = -\frac{\pi}{2}\ln 2$$

and $J = I$*.*

5.6 Exercises of the Chapter

Exercise 5.6.1 *Study the nature of the following integrals:*

$$\textbf{a)} \int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}dx \qquad \textbf{b)} \int_{-2}^{2} \frac{1}{x^2}dx \qquad \textbf{c)} \int_{-\infty}^{0} 2^r dr \qquad \textbf{d)} \int_{-\infty}^{\infty} \left(y^3 - 3y^2\right) dy$$

$$\textbf{e)} \int_{-\infty}^{\infty} \cos \pi t dt \qquad \textbf{f)} \int_0^1 \frac{\ln x}{\sqrt{x}}dx \qquad \textbf{g)} \int_0^{\infty} \frac{e^x}{e^{2x}+3}dx \qquad \textbf{h)} \int_0^5 \frac{w}{w-2}dw.$$

Correction 5.6.1 1. *Rewrite:*

$$\int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}dx = \lim_{t\to\infty} \int_0^t \frac{1}{\sqrt[4]{1+x}}dx = \lim_{t\to\infty} \int_0^t (1+x)^{-1/4}dx = \lim_{t\to\infty} \frac{4}{3}(1+x)^{3/4}\Big|_0^t$$
$$\lim_{t\to\infty} \frac{4}{3}(1+t)^{3/4} - \frac{4}{3} = \infty$$

So the integral diverges.

2. *There are two ways to do this problem, so I will post both solutions.*

One way: Split up the integral at $x = 0$ *:*

$$\int_{-2}^{2} \frac{1}{x^2}dx = \int_{-2}^{0} \frac{1}{x^2}dx + \int_0^2 \frac{1}{x^2}dx = \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^2}dx + \lim_{s\to 0^+} \int_s^2 \frac{1}{x^2}dx$$
$$= \lim_{t\to 0^-} \frac{-1}{x}\Big|_{-2}^{t} + \lim_{s\to 0^+} \frac{-1}{x}\Big|_s^2 = \lim_{t\to 0^-} \left(\frac{-1}{t}\right) - \frac{1}{2} - \frac{1}{2} + \lim_{s\to 0^+} \left(\frac{1}{s}\right)$$

Both of the limits diverge so the integral diverges.
Another way: $\dfrac{1}{x^2}$ *is an even function, so it is symmetric about* $x = 0$ *:*

$$\int_{-2}^{2} \frac{1}{x^2} dx = 2\int_{0}^{2} \frac{1}{x^2} dx = \lim_{t\to 0^+} 2\int_{t}^{2} \frac{1}{x^2} dx = \lim_{t\to 0^+} 2\left(\frac{-1}{x}\right)\Big|_t^2 = -1 + 2\lim_{t\to 0^+}\frac{1}{t} = \infty$$

So the integral diverges.

3. *Rewrite:*

$$\int_{-\infty}^{0} 2^r dr = \lim_{t\to -\infty}\int_{t}^{0} 2^r dr = \lim_{t\to -\infty}\left(\frac{2^r}{\ln 2}\Big|_t^0\right) = \frac{1}{\ln 2} - \lim_{t\to -\infty}\left(\frac{2^t}{\ln 2}\right) = \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}$$

Convergent!

4. *Need to split it up, try about* $y = 0$ *:*

$$\int_{-\infty}^{\infty} \left(y^3 - 3y^2\right) dy = \int_{-\infty}^{0} \left(y^3 - 3y^2\right) dy + \int_{0}^{\infty} \left(y^3 - 3y^2\right) dy$$

$$= \lim_{t\to -\infty}\int_{t}^{0} \left(y^3 - 3y^2\right) dy + \lim_{s\to\infty}\int_{0}^{s} \left(y^3 - 3y^2\right) dy = \lim_{t\to -\infty}\left(\frac{y^4}{4} - y^3\right)\Big|_t^0 + \lim_{s\to\infty}\left(\frac{y^4}{4} - y^3\right)\Big|_0^s$$

$$= -\lim_{t\to -\infty}\left(\frac{t^4}{4} - t^3\right) + \lim_{s\to\infty}\left(\frac{s^4}{4} - s^3\right)$$

Both of these limits diverge, so the integral diverges.

5. *Need to split it up, try about* $t = 0$ *:*

$$\int_{-\infty}^{\infty} \cos\pi t dt = \int_{-\infty}^{0} \cos\pi t dt + \int_{0}^{\infty} \cos\pi t dt = \lim_{s\to -\infty}\int_{s}^{0} \cos\pi t dt + \lim_{r\to\infty}\int_{0}^{r} \cos\pi t dt$$

$$= \lim_{s\to -\infty}\left(\frac{1}{\pi}\sin\pi t\right)\Big|_s^0 + \lim_{r\to\infty}\left(\frac{1}{\pi}\sin\pi t\right)\Big|_0^r = -\lim_{s\to -\infty}\left(\frac{1}{\pi}\sin\pi s\right) + \lim_{r\to\infty}\left(\frac{1}{\pi}\sin\pi r\right)$$

Both of these limits diverge, so the integral diverges.

6. *Try a u-substitution first. Let* $u = \sqrt{x}, du = \dfrac{1}{2\sqrt{x}}dx \Rightarrow 2du = \dfrac{1}{\sqrt{x}}dx$*. When* $x = 0, u = 0$ *and when* $x = 1, u = 1$ *:*

$$\int_{0}^{1} \frac{\ln x}{\sqrt{x}} dx = \int_{0}^{1} \frac{\ln\left(\sqrt{x}^2\right)}{\sqrt{x}} dx = \int_{0}^{1} \ln\left(u^2\right) du = 2\int_{0}^{1} \ln u du$$

This is still improper because $\ln u$ *is undefined at* $u = 0$*. Rewrite with a limit:*

$$2\int_{0}^{1} \ln u du = \lim_{t\to 0^+} 2\int_{t}^{1} \ln u du$$

Use integration by parts (we did $\int \ln x dx$ in class once upon a time...):

$$= \lim_{t\to 0^+} 2(u\ln u - u)\Big|_t^1 = -2 - \lim_{t\to 0^+} 2(t\ln t - t) = -2 - 2\lim_{t\to 0^+} t\ln t + 2\lim_{t\to 0^+} t$$
$$= -2 - 2\lim_{t\to 0^+} t\ln t + 0 = -2 - 2\lim_{t\to 0^+} t\ln t$$

The right limit is what we call indeterminate because if we take the limit we get something that looks like $0\cdot -\infty$, which is no bueno. So we need to use L'Hôpital's Rule :

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+} \frac{(\ln t)'}{\left(\frac{1}{t}\right)'} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to 0^+} \frac{-t^2}{t} = \lim_{t\to 0^+} -t = 0$$

This shows that our integral is convergent, and it converges to $-2-2\lim_{t\to 0^+} t\ln t = -2-0 = -2$.

7. *Let's do a u-substitution first. Let $u = e^x$, then $du = e^x dx$. When $x = 0, u = 1$ and when $x \to \infty, u \to \infty$:*

$$\int_0^\infty \frac{e^x}{e^{2x}+3}dx = \int_0^\infty \frac{e^x}{(e^x)^2+3}dx = \int_1^\infty \frac{1}{u^2+3}du = \lim_{t\to\infty}\int_1^t \frac{1}{u^2+3}du$$
$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_1^t = \lim_{t\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Convergent!

8. *Try a u-substitution first. Let $u = w-2$, then $w = u+2, du = dw$. When $w = 0, u = -2$, and when $w = 5, u = 3$:*

$$\int_0^5 \frac{w}{w-2}dw = \int_{-2}^3 \frac{u+2}{u}du = \int_{-2}^3 \left(1+\frac{2}{u}\right)du$$

The function $1+\dfrac{2}{u}$ is discontinuous at $u = 0$. Need to split up the integral:

$$\int_{-2}^3 \left(1+\frac{2}{u}\right)du = \int_{-2}^0 \left(1+\frac{2}{u}\right)du + \int_0^3 \left(1+\frac{2}{u}\right)du$$
$$= \lim_{t\to 0^-}\int_{-2}^t \left(1+\frac{2}{u}\right)du + \lim_{s\to 0^+}\int_s^3 \left(1+\frac{2}{u}\right)du = \lim_{t\to 0^-}(u+2\ln|u|)\Big|_{-2}^t + \lim_{s\to 0^+}(u+2\ln|u|)\Big|_s^3$$
$$= \lim_{t\to 0^-}(t+2\ln|t|) + 2 - 2\ln 2 + 3 + 2\ln 3 - \lim_{s\to 0^+}(s+2\ln|s|)$$

Both of the limits diverge, so the integral diverges.

Exercise 5.6.2 *Use the Comparison Theorem to decide if the following integrals are convergent or divergent.*

1.

$$\int_1^\infty \frac{1+e^{-x}}{x}dx$$

2.

$$\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$$

Correction 5.6.2 *1. Let's guess that this integral is divergent. That means we need to find a function smaller than* $\frac{1+e^{-x}}{x}$ *that is divergent. To make it smaller, we can make the top smaller or the bottom bigger. Let's make the top smaller:*

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x}$$

Then take the integral:

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} \ln x\Big|_1^t = \lim_{t\to\infty} \ln t = \infty$$

So the integral diverges. Since $\int_1^{\infty} \frac{1}{x} dx$ *diverges, then* $\int_1^{\infty} \frac{1+e^{-x}}{x} dx$ *diverges.*

2. Let's guess that this integral is convergent. That means we need to find a function bigger than $\frac{\sin^2 x}{\sqrt{x}}$ *that is convergent. To make it bigger, we can make the top bigger or the bottom smaller. Let's make the top bigger:*

$$\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

Then take the integral:

$$\int_0^{\pi} \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} \int_t^{\pi} \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^{\pi} = 2\sqrt{\pi} - \lim_{t\to 0^+} \sqrt{t} = 2\sqrt{\pi} - 0 = 2\sqrt{\pi}$$

So the integral converges. Since $\int_0^{\pi} \frac{1}{\sqrt{x}} dx$ *converges, then* $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}} dx$ *converges.*

Exercise 5.6.3

1. *Study the nature of the following integrals depending on the values of* α*:*

$$\int_1^{+\infty} \frac{1}{t^{\alpha}}\,\mathrm{d}t \quad \text{and} \quad \int_0^1 \frac{1}{t^{\alpha}}\,\mathrm{d}t,$$

where $\alpha \in \mathbb{R}$.

2. *Using the comparison criterion, study the nature of the following integrals:*

$$\int_1^{+\infty} \frac{dt}{1+t^3} \quad \text{and} \quad \int_0^1 \frac{e^t+1}{t}\,\mathrm{d}t.$$

Correction 5.6.3 *1. The calculation of the primitive is explicit:*

$$\int_1^{+\infty} \frac{1}{t^{\alpha}}\,\mathrm{d}t = \begin{cases} \lim_{x\to+\infty} \left[\frac{1}{-\alpha+1}\frac{1}{t^{\alpha-1}}\right]_1^x & \text{if } \alpha \neq 1 \\ \lim_{x\to+\infty} [\ln t]_1^x & \text{if } \alpha = 1 \end{cases}$$

From this, we immediately deduce the nature (convergent or divergent) of the Riemann integrals:

$$\text{If } \alpha > 1, \quad \int_1^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t \quad \text{converges.}$$
$$\text{If } \alpha \leq 1, \quad \int_1^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t \quad \text{diverges.}$$

Similarly, we show that

$$\int_0^1 \frac{1}{t^\alpha}\,\mathrm{d}t = \begin{cases} \lim\limits_{x\to 0}\left[\frac{1}{-\alpha+1}\frac{1}{t^{\alpha-1}}\right]_x^1 & \text{if } \alpha \neq 1 \\ \lim\limits_{x\to 0}[\ln t]_x^1 & \text{if } \alpha = 1 \end{cases}$$

and we have

$$\text{If } \alpha < 1, \quad \int_0^1 \frac{1}{t^\alpha}\,\mathrm{d}t \quad \text{converges.}$$
$$\text{If } \alpha \geq 1, \quad \int_0^1 \frac{1}{t^\alpha}\,\mathrm{d}t \quad \text{diverges.}$$

2. (a) *We have*

$$\forall t \geq 1, \quad \frac{1}{1+t^3} \leq \frac{1}{t^3}.$$

Since $\int_0^1 \frac{1}{t^3}\,\mathrm{d}t$ *converges (Riemann integral with* $\alpha = 3 > 1$*), the comparison theorem for positive functions allows us to conclude that* $\int_0^1 \frac{1}{1+t^3}\,\mathrm{d}t$ *converges.*

(b) *We have*

$$\forall t \in \mathbb{R}_+^*, \quad \frac{e^t+1}{t} > \frac{1}{t}.$$

As $\int_0^1 \frac{1}{t}\,\mathrm{d}t$ *diverges (Riemann integral with* $\alpha = 1$*), the comparison theorem for positive functions allows us to conclude that* $\int_0^1 \frac{e^t+1}{t}\,\mathrm{d}t$ *diverges.*

Exercise 5.6.4 *Study the absolute convergence or the semi-convergence of the following integrals:*

a) $\displaystyle\int_2^{+\infty} \frac{\sin t}{t\ln t}\,dt$

b) $\displaystyle\int_2^{+\infty} \frac{\cos 2t}{t(\ln t)^\alpha}\,dt \quad (\alpha > 0)$

c) $\displaystyle\int_0^{+\infty} \frac{\sin t}{t^\alpha(1+t^2)}\,dt \quad (\alpha \in \mathbb{R})$

Correction 5.6.4 a) *The function* $t \mapsto \frac{1}{t\ln(t)}$ *is continuous, positive, and decreasing over the interval* $[2, +\infty[$*, and*

$$\lim_{t\to+\infty} \frac{1}{t\ln(t)} = 0.$$

Moreover, for any $A > 2$,

$$\left|\int_2^A \sin t\,dt\right| = |\cos 2 - \cos A| \leq 2.$$

By applying Abel's rule, the integral

$$\int_{2}^{+\infty} \frac{\sin t}{t\ln(t)}\,dt$$

is convergent. It is semi-convergent, because the integral

$$\int_{2}^{+\infty} \frac{|\sin t|}{t\ln(t)}\,dt$$

DV.

b) *We have:*

$$\left|\frac{\cos 2t}{x(\ln(x))^{\alpha}}\right| \le \frac{1}{x(\ln(x))^{\alpha}}.$$

Or:

$$\int_{2}^{+\infty} \frac{dx}{x(\ln(x))^{\alpha}} \quad \textit{is absolutely convergent if and only if } \alpha > 1.$$

Thus, the proposed integral converges absolutely if $\alpha > 1$.

For $0 < \alpha \le 1$, *Abel's rule can be used to prove convergence. Let:*

$$u_n = \int_{n\pi}^{(n+1)\pi} \frac{|\cos 2t|}{x(\ln(x))^{\alpha}}\,dx.$$

Then:

$$u_n \ge \frac{\textit{constant}}{(n+1)(\ln(n+1))^{\alpha}} \int_0^{2\pi} |\cos t|\,dt \sim \frac{M}{n(\ln(n))^{\alpha}} \quad \textit{as } n \to +\infty.$$

We see that $\sum u_n$ *diverges. Therefore, there is no absolute convergence.*

c) *The function* $t \mapsto f(t) = \dfrac{\sin t}{t^{\alpha}(1+t^2)}$ *is continuous over* $]0, +\infty[$.

Near $t = 0$:

$$f(t) \sim \frac{1}{t^{\alpha-1}} \quad \textit{so there is convergence if and only if } \alpha < 2.$$

As $t \to +\infty$: *Using the bound:*

$$\left|\frac{\sin t}{t^{\alpha}(1+t^2)}\right| \le \frac{1}{t^{\alpha+2}},$$

we conclude that there is absolute convergence if $\alpha > -1$.

Now, consider the case $\alpha \le -1$ *and apply Abel's rule. Let:*

$$g(t) = \frac{1}{t^{\alpha}(1+t^2)},$$

then:

$$g'(t) = \frac{t^2(\alpha+2)+\alpha}{t^{\alpha+1}(1+t^2)^2}.$$

Thus, if $\alpha + 2 > 0$, *we find:*

$$g'(t) < 0 \quad \textit{for } t > \left(\frac{|\alpha|}{2+\alpha}\right)^{1/2} = t_0.$$

Since:

$$\left|\int_a^b \sin t\,dt\right| \leq 2, \quad \forall a, b \in \mathbb{R},$$

we conclude that the integral converges.sing Abel's rule, we obtain the convergence of the integral:

$$\int_{t_0}^{+\infty} \frac{\sin t}{t^\alpha(1+t^2)}\,dt.$$

Case: $-2 < \alpha \leq -1$

Let us show that for $-2 < \alpha \leq -1$, the integral does not converge absolutely. Indeed, let $a > 0$:

$$\int_a^{+\infty} \frac{|\sin t|}{t^\alpha(1+t^2)}\,dt \geq \int_a^{+\infty} \frac{\sin^2 t}{t^\alpha(1+t^2)}\,dt = \int_a^{+\infty} \frac{1-\cos 2t}{2t^\alpha(1+t^2)}dt.$$

We see that:

$$\int_a^{+\infty} \frac{\cos 2t}{2t^\alpha(1+t^2)}\,dt \quad \textit{converges (using Abel's rule again),}$$

but:

$$\int_a^{+\infty} \frac{dt}{t^\alpha(1+t^2)} \quad \textit{diverges because } \alpha + 2 < 0.$$

Thus, $\displaystyle\int_a^{+\infty} \frac{\sin^2 t}{t^\alpha(1+t^2)}\,dt$ *diverges, and:*

$$\int_a^{+\infty} \frac{|\sin t|}{t^\alpha(1+t^2)}\,dt \quad \textit{also diverges.}$$

It remains to study the case $\alpha \leq -2$.

Case: $\alpha \leq -2$

Consider the integral:

$$I_n = \int_{u_n}^{v_n} \frac{\sin t}{t^\alpha(1+t^2)}\,dt,$$

where $u_n = 2n\pi + \dfrac{\pi}{4}$ and $v_n = 2n\pi + \dfrac{\pi}{2}$.

We have $\sin t \geq \dfrac{1}{\sqrt{2}}$ for $t \in [u_n, v_n]$, hence

$$I_n \geq \int_{u_n}^{v_n} \frac{dt}{t^\alpha(1+t^2)} \geq \frac{1}{\sqrt{2}} \int_{u_n}^{v_n} \frac{t^2}{1+t^2}\,dt.$$

Then:

$$I_n = \frac{1}{\sqrt{2}}\big[t - \arctan t\big]_{u_n}^{v_n} \to \frac{\pi}{4\sqrt{2}} \quad \textit{as } n \to +\infty.$$

Thus, I_n does not tend to zero as $n \to +\infty$, and the Cauchy criterion is not satisfied. Hence, the integral diverges.

Conclusion

$$\int_0^{+\infty} \frac{\sin t}{t^\alpha(1+t^2)}\,dt \quad \textit{is:}$$

$$\textit{Absolutely convergent if } -1<\alpha<2,$$

$$\textit{Convergent (but not absolutely) if } -2<\alpha\le -1.$$

Chapter 6

Integrals Depending on a Parameter

6.1 Introduction

In this chapter we consider functions of the following type of the variable x, the integration variable being always denoted t: More precisely, let f be a function defined on $A \times I$, where A and I are two intervals of $\mathbb{R}$ (with non-empty interiors), and with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We can then consider, when it makes sense, the function g defined for $x \in A$ and with values in $\mathbb{K}$ by:

$$g(x) = \int_I f(x, t)\, \mathrm{d}t.$$

Sufficient conditions for the existence of $g(x)$ will be specified later. This expression is called an integral depending on a parameter. The questions that may arise then concern the calculation of the limits of g at certain points, the continuity of g, its differentiability and the calculation of its derivative, the calculation of an integral of g.
The previous notations will be preserved in the rest of the chapter.

6.2 The Dominated Convergence Theorem

Theorem 6.2.1 (The Dominated Convergence Theorem, or Lebesgue's Theorem) *Let $(f_n)n \in \mathbb{N}$ be a sequence of functions from I to $\mathbb{K}$. We assume*

1) $\forall n \in \mathbb{N}$ f_n is piecewise continuous.

2) The sequence (f_n) converges pointwise on I to a certain function f.

3) f is piecewise continuous.

4) There exists a function φ, piecewise continuous, such that $\int_I \varphi$ is convergent, and such that

$$\forall t \in I,\ \forall n \in \mathbb{N},\ |f_n(t)| \leq \varphi(t) \quad \textit{(Domination hypothesis.)}$$

Then the f_n and f are integrable on I, and

$$\lim_{n \to \infty} \int_I f_n = \int_I f.$$

Examples 6.2.2

1. *For $n \in \mathbb{N}$, we set:*

$$I_n = \int_0^{+\infty} \underbrace{\frac{\sin(n^2 t)}{1 + nt^2}}_{f_n(t)}\, dt.$$

1) *The functions* f_n *are continuous on* $\mathbb{R}^+$.

2) *The* f_n *are continuous and converge pointwise to the zero function, which is itself continuous. Indeed,*

$$\lim_{n\to\infty} \frac{\sin(n^2t)}{1+nt^2} = \begin{cases} 0 \ \textit{if}\ t=0 \\ 0 \ \textit{if}\ t\neq 0. \end{cases}$$

3)

$$\forall t,\ \forall n\geq 1,\quad |f_n(t)| \leq \frac{1}{1+t^2}.$$

And

$$\int_0^{+\infty} \frac{1}{1+t^2}dt \quad \textit{converges.}$$

Then by the dominated convergence theorem we have:

$$\lim_{n\to\infty} I_n = \int_0^{+\infty} 0 = 0.$$

2. *Let* f *be a continuous function on* $[0,1]$, *we set:*

$$J_n = \int_0^{+\infty} \underbrace{f(t^n)}_{g_n(t)}\, dt.$$

1) *The functions* g_n *are continuous on* $[0,1]$ $\forall n$.

2) *The sequence* (g_n) *converges pointwise to the piecewise continuous function:*

$$\lim_{n\to\infty} g_n(t) = h(t) = \begin{cases} f(0) \ \textit{if}\ 0\leq t<1 \\ f(1) \ \textit{if}\ t=1. \end{cases}$$

3) *Let* M *be an upper bound for the continuous function* $|f|$ *on* $[0,1]$. *Then, we have:*

$$|f(x)| \leq M, \quad \forall x\in[0,1].$$

Thus

$$\forall t,\ \forall n,\quad |g_n(t)| \leq M.$$

And

$$\int_0^1 M dt \quad \textit{converges.}$$

Then, by the Dominated Convergence Theorem, we obtain:

$$\lim_{n\to\infty} J_n = \int_0^1 h(t)dt = f(0).$$

6.3 Functions Defined by an Integral

We consider here functions of the following form:

$$g(x) = \int_I f(x,t)dt.$$

Here, we focus on the continuity of such functions and the possibility of differentiating them. Let us fix a and examine the problem of the continuity of g at a:

$$\lim_{x\to a} g(x) \stackrel{?}{=} g(a).$$

Or again:

$$\lim_{x\to a}\int_I f(x,t)dt \stackrel{?}{=} \int_I f(a,t)dt.$$

Remark 6.3.1 *We will, when the time comes, differentiate the function of two variables $(x,t)\longmapsto f(x,t)$ with respect to the variable x, which has not yet been studied in the Analysis 3 course (but will be covered in the Analysis 4 course, in the chapter "Functions of Several Variables"). The theory may be delicate, but the practical steps are not. Differentiate with respect to the variable x the function $g : x \longmapsto f(x,t)$. More precisely, to obtain this partial derivative at (x_0,t_0), we fix t equal to t_0 and calculate:*

$$g'(x_0) = \lim_{x\to x_0}\frac{f(x,t_0)-f(x_0,t_0)}{x-x_0}.$$

By subsequently varying (x_0,t_0), the result obtained depends on the variables x and t, thereby defining a function of two variables called the partial derivative of f with respect to x, denoted by $\dfrac{\partial f}{\partial x}$. Thus, for all (x,t), we have:

$$g'(x) = \frac{\partial f}{\partial x}(x,t).$$

For example, if for all $(x,t)\in\mathbb{R}\times]0,+\infty[$, $f(x,t)=t^xe^{-t}$, then for all $(x,t)\in\mathbb{R}\times]0,+\infty[$, $\dfrac{\partial f}{\partial x}(x,t) = (\ln t)t^xe^{-t}$ and $\dfrac{\partial f}{\partial t}(x,t)=(x-t)t^{x-1}e^{-t}$.

6.3.1 Continuity of parameter dependent integral

Theorem 6.3.2 (Continuity of parameter dependent integral) *Let f be a function defined on $A\times I$, where A and I are real intervals, with values in $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. We assume that:*

a) $\forall\, x\in A$, $t\mapsto f(x,t)$ is piecewise continuous on I;

b) $\forall\, t\in I$, the function $x\mapsto f(x,t)$ is continuous on A;

c) There exists a function φ, piecewise continuous on I, with values in $\mathbb{R}_+$, and integrable on I such that:

$$\forall\,(x,t)\in A\times I\ ,\ \big|f(x,t)\big|\leqslant\varphi(t)\,\text{(domination hypothesis)}.$$

Then: the function

$$g\ :\ x\mapsto\int_I f(x,t)\,\mathrm{d}t \text{ is continuous on } A.$$

Remark 6.3.3

1. *Assumption c) implies that the function $t\mapsto f(x,t)$ is integrable on I. Therefore, the definition of g is well-defined. .*

2. *The result of this theorem remains valid if we assume that the domination hypothesis is satisfied for $(x,t)\in K\times I$, where K is any segment contained in A (or any other type of interval whose union covers A).*

Proof: Let $a \in A$. We can always find a segment K containing a and included in A, and we will assume that the domination hypothesis is satisfied for $(x,t) \in K \times I$.

According to the theorem on the sequential characterization of continuity, to show that g is continuous at a, it suffices to prove that, for any sequence (a_n) of elements of K that converges to a, the sequence $(g(a_n))$ converges to $g(a)$.

So let (a_n) be such a sequence. Let us set $f_n(t) = f(a_n, t)$, so that $g(a_n) = \displaystyle\int_I f_n(t)\ \mathrm{d}t$.

According to hypothesis a), the functions f_n are piecewise continuous on I.

According to hypothesis b), the sequence (f_n) converges pointwise on I to the function $h\colon t \mapsto f(a,t)$, and h is piecewise continuous on I.

Finally, hypothesis c) implies that: $\forall t \in I, \forall n \in \mathbb{N},\ |f_n(t)| \leqslant \varphi(t)$, with φ piecewise continuous and integrable on I.

The hypotheses of the *dominated convergence theorem* for the sequence of functions (f_n) are therefore satisfied.

This theorem then allows us to conclude that:

$$\lim_{n\to\infty} \int_I f_n(t)\ \mathrm{d}t = \int_I f(a,t)\ \mathrm{d}t \quad \text{or equivalently:} \quad \lim_{n\to\infty} g(a_n) = g(a).$$

□

Remark 6.3.4 *The following example shows that the domination hypothesis is essential:*

Example 6.3.5 *The function* $f : (x,t) \mapsto \dfrac{x}{1+x^2t^2}$ *is continuous on* $\mathbb{R}^2$*, and for all* x*, the function* $t \mapsto \dfrac{x}{1+x^2t^2}$ *is integrable on* $\mathbb{R}_+$*.*
If we define, for all $x \in \mathbb{R}$*,* $g(x) = \displaystyle\int_0^{+\infty} f(x,t)\ \mathrm{d}t$*, we then obtain, by a simple calculation:*

$$\begin{aligned} g(0) &= 0, \\ g(x) &= \frac{\pi}{2} \quad \textit{if} \quad x > 0, \\ g(x) &= -\frac{\pi}{2} \quad \textit{if} \quad x < 0, \end{aligned}$$

and g *is not continuous at* 0*!*
(However, we can verify that the domination hypothesis is indeed satisfied on any segment $[a,b]$ contained in $\mathbb{R}_+$*, which implies continuity on* $\mathbb{R}_+$*.)*

Example 6.3.6 *We define*

$$g(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t^2}\,dt.$$

For fixed x*, the integrand is continuous on* $\mathbb{R}^+$*.*
If $x < 0$*, the integrand tends to* $+\infty$*, and the integral diverges.*
If $x \geq 0$*, we have*

$$0 \leq \frac{e^{-xt}}{1+t^2} \leq \frac{1}{t^2} \quad \textit{and} \quad \int_1^{+\infty} \frac{1}{t^2}\,dt \quad \textit{converges.}$$

Therefore, the domain of definition of g *is* $\mathbb{R}^+$*.*
Next, we observe:

- *For all $x \in \mathbb{R}^+$, $f(x, \cdot)$ is piecewise continuous.*
- *For all $t \in \mathbb{R}^+$, $f(\cdot, t)$ is continuous.*
- *For all $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$,*

$$|f(x,t)| = \left|\frac{e^{-xt}}{1+t^2}\right| \leq \frac{1}{1+t^2}, \quad \text{which is an integrable function independent of } x.$$

Thus, the function g is continuous on $\mathbb{R}^+$.

6.4 Limit of an Integral with a Parameter

Theorem 6.4.1 (Dominated Convergence Theorem for a Continuous Parameter) *Let f be a function defined on $A \times I$, where A and I are real intervals, with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*
Let $a \in \overline{A}$. We assume that:

a) $\forall x \in A$, $t \mapsto f(x,t)$ is piecewise continuous on I;

b) $\forall t \in I$, $\lim\limits_{x \to a} f(x,t) = \ell(t)$, where ℓ is a piecewise continuous function on I;

c) There exists a function φ, piecewise continuous on I, with values in $\mathbb{R}_+$, and integrable on *I such that:*

$$\forall (x,t) \in A \times I \ , \ \big|f(x,t)\big| \leqslant \varphi(t) \, \textit{(domination hypothesis)}$$

Then: the function ℓ is integrable on I, and

$$\lim_{x \to a} \left(\int_I f(x,t)\, \mathrm{d}t \right) = \int_I \ell(t)\, \mathrm{d}t$$

Proof: By taking the limit in c), we have $|\ell(t)| \leq \varphi(t)$ for all $t \in I$, so ℓ is integrable on I. For the second result, it suffices to use the sequential characterization of the limit: by considering a sequence (x_n) of elements of A that tends to a, and by setting $f_n(t) = f(x_n, t)$, the *theorem* of dominated convergence applies (it's the same method as in the proof of *theorem* 6.3.2).

□

Example 6.4.2 *Calculate $\lim\limits_{x \to +\infty} \int_0^{+\infty} e^{-tx} \arctan(t)\, \mathrm{d}t$, then find an equivalent.*
Indeed, applying theorem *6.4.1 for the limit. For the equivalent, integrate twice by parts.*

6.5 Differentiation

Theorem 6.5.1 *Let f be a function defined on $A \times I$, where A and I are real intervals, with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We assume that:*

a) For all $x \in A$, the map $t \mapsto f(x,t)$ is piecewise continuous and integrable *on I.*

b) For all $t \in I$, the function $x \mapsto f(x,t)$ is of class C^1 on A.

c) For all $x \in A$, the function $t \mapsto \dfrac{\partial f}{\partial x}(x,t)$ is piecewise continuous on I.

d) There exists a function φ, piecewise continuous on I, with values in $\mathbb{R}_+$, and integrable on I, such that

$$\forall (x,t) \in A \times I, \quad \left|\frac{\partial f}{\partial x}(x,t)\right| \leqslant \varphi(t) \quad \textit{(domination hypothesis)}.$$

Then, the function $g : x \mapsto \int_I f(x,t)\ \mathrm{d}t$ *is of class* C^1 *on* A*, and*

$$\forall x \in A, \quad g'(x) = \int_I \frac{\partial f}{\partial x}(x,t)\ \mathrm{d}t.$$

Remark 6.5.2

1. *Assumption d) implies that* $t \mapsto \frac{\partial f}{\partial x}(x,t)$ *is indeed integrable on* I*.*
2. *The result of this theorem remains true if we only assume that the domination hypothesis is verified for* $(x,t) \in K \times I$*, for any segment* $K \subset A$ *(or for any other type of interval whose union is* A*).*

Proof: Let $x_0 \in A$, and (h_n) be any sequence tending to 0 ($h_n \neq 0$). We have:

$$\frac{g(x_0+h_n) - g(x_0)}{h_n} = \int_I \frac{f(x_0+h_n,t) - f(x_0,t)}{h_n}\ \mathrm{d}t$$

. Let us define

$$f_n(t) = \frac{f(x_0+h_n,t) - f(x_0,t)}{h_n}.$$

According to hypothesis a), the f_n are piecewise continuous and integrable on I.
According to hypothesis b), the sequence of functions (f_n) converges pointwise on I to the function $t \mapsto \frac{\partial f}{\partial x}(x_0,t)$, which is piecewise continuous according to hypothesis c).
We also have (cf. results on primitives...)

$$f_n(t) = \frac{1}{h_n}\int_{x_0}^{x_0+h_n} \frac{\partial f}{\partial x}(x,t)\ \mathrm{d}x.$$

Therefore, using hypothesis d), we deduce that for all $t \in I$, for all $n \in \mathbb{N}$, $|f_n(t)| \leqslant \varphi(t)$. The hypotheses of the *dominated convergence theorem* for the sequence of functions (f_n) are therefore satisfied. This theorem then allows us to assert that:

$$\lim_{n\to\infty} \int_I f_n(t)\ \mathrm{d}t = \int_I \frac{\partial f}{\partial x}(x_0,t)\ \mathrm{d}t$$

or again:

$$\lim_{n\to\infty} \frac{g(x_0+h_n) - g(x_0)}{h_n} = \int_I \frac{\partial f}{\partial x}(x_0,t)\ \mathrm{d}t.$$

This being true for any sequence (h_n), the theorem on the sequential characterization of the limit gives:

$$\lim_{h\to 0, h\neq 0} \frac{g(x_0+h) - g(x_0)}{h} = \int_I \frac{\partial f}{\partial x}(x_0,t)\ \mathrm{d}t,$$

which proves that g is differentiable at x_0 and that $g'(x_0) = \int_I \frac{\partial f}{\partial x}(x_0,t)\ \mathrm{d}t$.
Finally, the continuity of g' results from applying Theorem 6.3.2 to $\frac{\partial f}{\partial x}$, further using hypothesis b).

□

A recursion (not so simple) allows to generalize the previous theorem to functions of class C^n:

Corollary 6.5.3 (Generalization) *Let* f *be a function defined on* $A \times I$*, where* A *and* I *are real intervals, with values in* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. Let* $n \in \mathbb{N}^*$*. We assume that:*

a) $f, \dfrac{\partial f}{\partial x}, \ldots, \dfrac{\partial^n f}{\partial x^n}$ *exist, are continuous with respect to* x *on* A*, and piecewise continuous with respect to* t *on* I*.*

b) For all $x \in A$*, the functions* $t \mapsto f(x,t)$*,* $t \mapsto \dfrac{\partial f}{\partial x}(x,t)$*, ...,* $t \mapsto \dfrac{\partial^{n-1} f}{\partial x^{n-1}}(x,t)$ *are integrable on* I*.*

c) There exists a function φ*, piecewise continuous on* I*, with values in* $\mathbb{R}^+$*, and* integrable on I *such that:*

$$\forall (x,t) \in A \times I, \quad \left|\frac{\partial^n f}{\partial x^n}(x,t)\right| \leqslant \varphi(t) \quad \textit{(domination hypothesis).}$$

Then, the function $g : x \mapsto \displaystyle\int_I f(x,t)\ \mathrm{d}t$ *is of class* C^n *on* A*, and:*

$$\forall k \in \mathbb{N}^*,\ \forall x \in A, \quad g^{(k)}(x) = \int_I \frac{\partial^k f}{\partial x^k}(x,t)\ \mathrm{d}t.$$

Remark 6.5.4 *The result of this theorem remains true if we only assume that the domination hypothesis is verified for* $(x,t) \in K \times I$*, for any segment* $K \subset A$ *(or for any other type of interval whose union is* A*).*

Proof: The result at order 1 is nothing other than the previous derivation theorem.
Suppose the result is verified at order n, and let f satisfy the above hypotheses at order $n+1$. In particular, there exists a function φ, piecewise continuous on I, with values in $\mathbb{R}_+$ and *integrable on* I such that:

$$\forall (x,t) \in A \times I, \ \left|\frac{\partial^{n+1} f}{\partial x^{n+1}}(x,t)\right| \leqslant \varphi(t)$$

If $K = [a,b]$ is a segment included in A, and if $t \in I$ is fixed, the mean value theorem applied to the function $x \mapsto \dfrac{\partial^n f}{\partial x^n}(x,t)$ on K gives:

$$\begin{aligned}\forall x \in K, \ \left|\frac{\partial^n f}{\partial x^n}(x,t) - \frac{\partial^n f}{\partial x^n}(a,t)\right| &\leq (b-a)\left|\frac{\partial^{n+1} f}{\partial x^{n+1}}(x,t)\right| \\ &\leq (b-a)\varphi(t),\end{aligned}$$

whence:

$$\forall (x,t) \in K \times I, \ \left|\frac{\partial^n f}{\partial x^n}(x,t)\right| \leq \left|\frac{\partial^n f}{\partial x^n}(a,t)\right| + (b-a)\varphi(t)$$

and the right-hand side of this inequality is an integrable function on I.
Thus, f indeed satisfies the hypotheses of the theorem at rank n. According to the induction hypothesis, we deduce that g is of class C^n on A and that

$$\forall x \in A, \ g^{(n)}(x) = \int_I \frac{\partial^n f}{\partial x^n}(x,t)\, dt.$$

It only remains to apply the derivation theorem to the parameterized integral above to obtain the result at order $n+1$.

□

Corollary 6.5.5 (Generalization) *Let* f *be a function defined on* $A \times I$*, where* A *and* I *are real intervals, with values in* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. Let* $n \in \mathbb{N}$*. We assume that:*

a) For all $n \in \mathbb{N}$, $\dfrac{\partial^n f}{\partial x^n}$ exists, is continuous with respect to x on A, and piecewise continuous with respect to t on I.

b) For all $n \in \mathbb{N}$, there exists a function φ_n, piecewise continuous on I, with values in $\mathbb{R}_+$ and integrable on I *such that:*

$$\forall (x,t) \in A \times I, \ \left|\frac{\partial^n f}{\partial x^n}(x,t)\right| \leqslant \varphi_n(t) \ \textit{(domination hypothesis)}.$$

Then, the function $g : x \mapsto \displaystyle\int_I f(x,t)\,dt$ is of class C^∞ on A, and:

$$\forall k \in \mathbb{N}, \ \forall x \in A, \ g^{(k)}(x) = \int_I \frac{\partial^k f}{\partial x^k}(x,t)\,dt.$$

Remark 6.5.6 *The result of this theorem remains true if we only assume that the domination hypothesis is verified for $(x,t) \in K \times I$, for any segment K included in A (or for any other type of interval whose union is A).*

Proof: It is clear that the hypotheses made in this 2nd corollary imply the hypotheses of the 1st corollary for all n.

□

Example 6.5.7 (The Gaussian integral) *We set, for $x \in \mathbb{R}$,*

$$F(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\,dt, \quad G(x) = \int_0^x e^{-t^2}\,dt.$$

G is of class C^1 on $\mathbb{R}$ and $\forall x \in \mathbb{R}$, $G'(x) = e^{-x^2}$. Let:

$$f : \begin{cases} \mathbb{R} \times [0,1] \longrightarrow \mathbb{R} \\ (x,t) \longmapsto \dfrac{e^{-x^2(1+t^2)}}{1+t^2} \end{cases}$$

- *f is piecewise continuous with respect to t.*
- *$\displaystyle\int_0^1 f(x,t)\,dt$ converges $\forall x \in \mathbb{R}$ (it is proper).*
- *$\dfrac{\partial f}{\partial x}$ exists at every point $\mathbb{R} \times [0,1]$. Furthermore, $\dfrac{\partial f}{\partial x}(x,t) = -2xe^{-x^2(1+t^2)}$ is piecewise continuous with respect to t, continuous with respect to x, and we have*

$$\begin{aligned} \left|\frac{\partial f}{\partial x}(x,t)\right| &\leq 2\,|x|\,e^{-x^2} \quad \textit{(bounded function on)}\ \mathbb{R} \\ &\leq M \quad \textit{(integrable on)}\ [0,1]. \end{aligned}$$

Then the function F is of class C^1 and can be derived under the integral sign $\displaystyle\int$:

$$\begin{aligned} F'(x) &= \int_0^1 -2xe^{-x^2(1+t^2)}\,dt \\ &= -2xe^{-x^2}\int_0^1 e^{-x^2t^2}\,dt \\ &= -2e^{-x^2}\int_0^x e^{-u^2}\,du \\ &= -2G'(x)G(x). \end{aligned}$$

Therefore, $\forall x \in \mathbb{R}$, there exists a constant C such that

$$F(x) = C - G^2(x).$$

For $x = 0$:

$$F(0) = \int_0^1 \frac{dt}{1+t^2} = \frac{\pi}{4}, \quad G(0) = 0.$$

Then, $\forall x \in \mathbb{R}$:

$$F(x) = \frac{\pi}{4} - G^2(x). \tag{6.1}$$

But

$$0 \leq F(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\, dt \leq e^{-x^2} \underset{x \to +\infty}{\longrightarrow} 0.$$

And

$$\lim_{x \to +\infty} G(x) = \int_0^{+\infty} e^{-t^2}\, dt.$$

Therefore, when we let x tend to $+\infty$ in (6.1), we get

$$\left(\int_0^{+\infty} e^{-t^2}\, dt \right)^2 = \frac{\pi}{4}.$$

Hence

$$\int_0^{+\infty} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

We will apply Theorem 6.5.1 in the following paragraph, where we study Euler's Γ function.

6.6 Definition and Study of the Gamma Function Γ

Theorem 6.6.1 (The Γ function) *For all $x > 0$, we can define the function*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt.$$

This function is of class C^∞ on $\mathbb{R}^+$ and, for all $n \in \mathbb{N}$, we have

$$\Gamma^{(n)}(x) = \int_0^{+\infty} e^{-t} (\ln t)^n t^{x-1}\, dt.$$

Proof:

- Existence:

  Let's set, for $x > 0$ and $t \in \mathbb{R}^+$, $f(x,t) = \mathrm{e}^{-t} t^{x-1}$. f is obviously continuous. Moreover, for all $x > 0$, we have $f(x,t) \underset{t \to 0^+}{\sim} t^{x-1}$, and the function $t \mapsto t^{x-1}$ is integrable in the neighborhood of 0 (Riemann integral). Also, for all $x > 0$, we have $f(x,t) \underset{t \to +\infty}{=} o\left(\frac{1}{t^2}\right)$, and the function $t \mapsto \frac{1}{t^2}$ is integrable in the neighborhood of $+\infty$. This ensures the existence of $\Gamma(x)$ for $x > 0$.

- Continuity:

  Let's set, for $x > 0$, $\Gamma(x) = g_1(x) + g_2(x)$, with $g_1(x) = \displaystyle\int_0^1 \mathrm{e}^{-t} t^{x-1}\,dt$ and $g_2(x) = \displaystyle\int_1^{+\infty} \mathrm{e}^{-t} t^{x-1}\,dt$.
  Let $a, b \in \mathbb{R}$ such that $0 < a < b$. The function $(x,t) \mapsto f(x,t)$ is continuous on $[a,b] \times \mathbb{R}^+$.
  For $0 < t \leqslant 1$, we have $0 \leqslant f(x,t) \leqslant \mathrm{e}^{-t} t^{a-1}$, where the function $t \mapsto \mathrm{e}^{-t} t^{a-1}$ is integrable on $]0,1]$. Therefore, g_1 is continuous on $[a,b]$.
  For $t \geqslant 1$, we have $0 \leqslant f(x,t) \leqslant \mathrm{e}^{-t} t^{b-1}$, where the function $t \mapsto \mathrm{e}^{-t} t^{b-1}$ is integrable on $[1,+\infty[$. Therefore, g_2 is continuous on $[a,b]$.

  Thus, Γ is continuous on any interval $[a,b]$ with $0 < a < 1 < b$, and therefore on $\mathbb{R}^+$.

- Differentiability:

  For all $n \in \mathbb{N}$, we have: $\dfrac{\partial^n f}{\partial x^n} = \mathrm{e}^{-t} (\ln t)^n t^{x-1}$. We show, exactly as above (but using Bertrand integrals in the neighborhood of 0), that Γ is of class $\mathcal{C}^n$ on $\mathbb{R}^+$, and that

$$\Gamma^{(n)}(x) = \int_0^{+\infty} \mathrm{e}^{-t} (\ln t)^n t^{x-1}\,dt.$$

□

Proposition 6.6.2 (Properties of the Γ function) *The Γ function satisfies the following properties:*

1. $\forall\, x > 0,\ \Gamma(x+1) = x\Gamma(x)$.
2. $\forall\, n \in \mathbb{N}^*,\ \Gamma(n) = (n-1)!$.
3. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

Proof:

1. Integration by parts
2. Recurrence
3. cf. Gaussian integral, after change of variable.

□

Proposition 6.6.3 (Study at the boundaries)

1. *We have:* $\lim\limits_{x\to 0^+} \Gamma(x) = +\infty$. *More precisely,* $\Gamma(x) \underset{x\to 0^+}{\sim} \dfrac{1}{x}$
2. $\lim\limits_{x\to+\infty} \Gamma(x) = +\infty$ *and* $\lim\limits_{x\to+\infty} \dfrac{\Gamma(x)}{x} = +\infty$

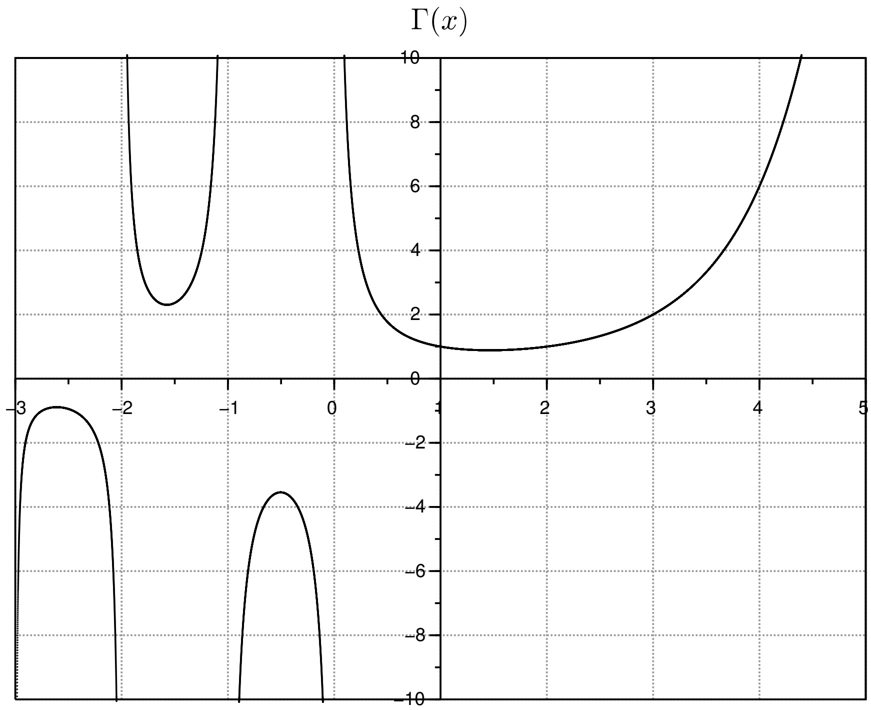
Proof:

1. The relation $x\Gamma(x) = \Gamma(x+1)$ gives $\lim\limits_{x \to 0^+} x\Gamma(x) = \Gamma(1) = 1$.

2.
   - Since Γ is of class C^1 and that $\Gamma(1) = \Gamma(2)$, the function Γ' vanishes at $\alpha \in]1, 2[$ according to Rolle's theorem. Since Γ' is strictly increasing, $\Gamma'(x) < 0$ for $x < \alpha$ and $\Gamma'(x) > 0$ for $x > \alpha$. In particular, Γ is increasing on $[\alpha, +\infty[$. According to the monotone convergence theorem, it therefore admits a limit in $\overline{\mathbb{R}}$ when x tends to $+\infty$. Since $\Gamma(n+1) = n!$ if $n \in \mathbb{N}$, this limit is $+\infty$.
   - For $x > 1$, $\dfrac{\Gamma(x)}{x} = \dfrac{x-1}{x}\Gamma(x-1)$, so $\lim\limits_{x \to +\infty} \dfrac{\Gamma(x)}{x} = +\infty$. This means that the curve representing Γ has a parabolic branch in direction Oy at $+\infty$.

$\square$

Graphical representation of the Γ function :



6.6.1 Complements on the Gamma function

1) We can prove that $\Gamma'(1) = \displaystyle\int_0^{+\infty} e^{-t} \ln t dt = -\gamma$ (γ Euler's constant). Therefore, in the neighborhood of 0,

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(1) + x\Gamma'(1) + o(x)}{x} = \frac{1}{x} - \gamma + o(1).$$

2) Γ is logarithmically convex i.e., $\ln \Gamma$ is convex. Indeed:

$$\begin{aligned}(\Gamma'(x))^2 &= \left(\int_0^{+\infty} e^{-t} \ln t t^{x-1} dt\right)^2 \\ &= \left(\int_0^{+\infty} e^{-\frac{t}{2}} t^{\frac{x-1}{2}} \ln t e^{-\frac{t}{2}} t^{\frac{x-1}{2}} dt\right)^2 \\ &\leq \int_0^{+\infty} \left(e^{-\frac{t}{2}} t^{\frac{x-1}{2}}\right)^2 dt \int_0^{+\infty} \left(e^{-\frac{t}{2}} t^{\frac{x-1}{2}} \ln t\right)^2 dt \\ &\leq \Gamma(x)\Gamma''(x).\end{aligned}$$

Therefore

$$(\ln \Gamma)'' = \frac{\Gamma''\Gamma - (\Gamma')^2}{\Gamma^2} \geq 0.$$

3) Stirling:

$$\Gamma(x+1) \underset{x\to+\infty}{\sim} \left(\frac{x}{e}\right)^x \sqrt{2\pi x}.$$

4) Complements formula

$$\forall x \in]0,1[, \ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

6.7 Exercises of the Chapter

Exercise 6.7.1 *Compute:*

$$g(x) = \int_0^1 \frac{\mathrm{d}t}{(t^2+x^2)^2}, \quad \textit{for } x \neq 0.$$

Correction 6.7.1 *The function* $f\colon (x,t) \mapsto \dfrac{1}{x^2+t^2}$ *is continuous on* $\mathbb{R}^* \times \mathbb{R}$*, and its partial derivative*

$$\frac{\partial f}{\partial x}(x,t) = \frac{-2x}{(x^2+t^2)^2}$$

is also continuous on $\mathbb{R}^* \times \mathbb{R}$*. Moreover, if* $[a,b]$ *is a segment included in* $\mathbb{R}_+^*$*, then*

$$\forall x \in [a,b], \ \forall t \in [0,1], \ \left|\frac{\partial f}{\partial x}(x,t)\right| \leq \frac{2b}{(a^2+t^2)^2} = \varphi(t),$$

where φ *is continuous and thus integrable on the segment* $[0,1]$*. (A similar result holds for a segment included in* $\mathbb{R}_-^*$*.)*

Thus, by defining $h(x) = \displaystyle\int_0^1 \frac{\mathrm{d}t}{x^2+t^2}$ *for* $x \in \mathbb{R}^*$*, the theorem covered in the course allows us to assert that* h *is of class* C^1 *on* $\mathbb{R}^*$ *and that:*

$$\forall x \neq 0, \quad h'(x) = \int_0^1 \frac{-2x \, \mathrm{d}t}{(x^2+t^2)^2} = -2x\, g(x).$$

Conversely,

$$h(x) = \frac{1}{x} \arctan\left(\frac{1}{x}\right),$$

which implies:

$$g(x) = \frac{-1}{2x} h'(x) = \frac{-1}{2x}\left[\frac{-1}{x^2} \arctan\left(\frac{1}{x}\right) - \frac{1}{x(x^2+1)}\right].$$

Thus:

$$g(x) = \frac{1}{2x^3} \arctan\left(\frac{1}{x}\right) + \frac{1}{2x^2(x^2+1)}.$$

Remark:
A similar reasoning enables us to compute, by induction on $n \geqslant 2$, the integrals

$$\int_0^1 \frac{\mathrm{d}t}{(t^2+x^2)^n}.$$

Exercise 6.7.2

Compute

$$g(x) = \int_0^{\frac{\pi}{2}} \ln(x\cos^2 t + \sin^2 t)\,\mathrm{d}t$$

for $x > 0$.

Correction 6.7.2 *Let us define $f(x,t) = \ln(x\cos^2 t + \sin^2 t)$ for $x > 0$ and $t \in \left[0, \frac{\pi}{2}\right]$.*

- *Differentiability on $\mathbb{R}_+^*$:*
  Initially, for any fixed $x > 0$, the function $t \mapsto f(x,t)$ is continuous on the interval $\left[0, \frac{\pi}{2}\right]$, because $x\cos^2 t + \sin^2 t$ remains strictly positive within this range.

  $$\frac{\partial f}{\partial x}(x,t) = \frac{\cos^2 t}{x\cos^2 t + \sin^2 t},$$

  hence $\frac{\partial f}{\partial x}(x,t)$ is defined and continuous on $\mathbb{R}_+^* \times \left[0, \frac{\pi}{2}\right]$.

  Then, for any segment $[a,b] \subset \mathbb{R}_+^*$,

  $$\forall x \in [a,b],\ \forall t \in \left[0, \frac{\pi}{2}\right],\ \left|\frac{\partial f}{\partial x}(x,t)\right| \leq \frac{\cos^2 t}{a\cos^2 t + \sin^2 t} = \varphi(t),$$

  where φ is continuous and therefore integrable on $\left[0, \frac{\pi}{2}\right]$.

  Based on the theorem from the course, g is C^1 on $\mathbb{R}_+^*$, and:

  $$\forall x > 0, \quad g'(x) = \int_0^{\frac{\pi}{2}} \frac{\cos^2 t}{x\cos^2 t + \sin^2 t}\,\mathrm{d}t.$$

  By substituting $u = \tan t$ (a strictly increasing C^1 bijection from $\left[0, \frac{\pi}{2}\right[$ to $[0, +\infty[$), we obtain:

  $$g'(x) = \int_0^{+\infty} \frac{\mathrm{d}u}{(1+u^2)(x+u^2)} = \frac{1}{x-1}\int_0^{+\infty}\left(\frac{1}{1+u^2} - \frac{1}{x+u^2}\right)\mathrm{d}u \quad \text{(for } x \neq 1\text{)}.$$

  $$= \frac{1}{x-1}\left[\arctan(u) - \frac{1}{\sqrt{x}}\arctan\left(\frac{u}{\sqrt{x}}\right)\right]_0^{+\infty} = \frac{\pi}{2}\frac{1}{x+\sqrt{x}},$$

  which also holds for $x = 1$ by the continuity of g'.

- *Value of g:*
  Thus, $\forall x > 0$, $g'(x) = \frac{\pi}{2}\frac{1}{x+\sqrt{x}}$ and $g(1) = 0$ (an obvious calculation), which gives:

  $$\forall x > 0, \quad g(x) = \frac{\pi}{2}\int_1^x \frac{\mathrm{d}t}{t+\sqrt{t}} = \pi\int_1^x \frac{\mathrm{d}t}{2\sqrt{t}(1+\sqrt{t})} = \pi\left[\ln(\sqrt{t}+1)\right]_1^x,$$

  and finally:

$$\boxed{\forall x > 0, \quad g(x) = \pi \ln\left(\frac{\sqrt{x}+1}{2}\right).}$$

- *Continuity of g at 0^+:*

  For this study, we will use the theorem from the course. The first two assumptions of this theorem are easily verified, and all that remains is to verify the domination assumption. It suffices to prove the continuity of g on $[0,1]$ (The continuity on $\mathbb{R}_+^*$ follows from its differentiability, as shown earlier).

  Suppose $x \in [0,1]$; $\forall t \in \left]0, \frac{\pi}{2}\right]$, $\sin^2 t \leq x\cos^2 t + \sin^2 t \leq 1$, hence:

  $$\forall x \in [0,1],\ \forall t \in \left]0, \frac{\pi}{2}\right],\ |f(x,t)| \leq \left|\ln(\sin^2 t)\right| = 2\left|\ln \sin t\right| = \varphi(t),$$

  where φ is continuous and integrable on $\left]0, \frac{\pi}{2}\right]$ (easy to verify).

  Thus, the hypothesis of domination is satisfied, and we conclude that g is continuous on $[0,1]$, hence on $\mathbb{R}_+$. In particular, g is continuous at 0^+, so:

  $$\lim_{x\to 0^+} g(x) = g(0) = \int_0^{\frac{\pi}{2}} \ln(\sin^2 t)\,\mathrm{d}t.$$

  From the earlier calculation of $g(x)$ for $x > 0$, this limit is equal to $-\pi\ln 2$. We can deduce that:

  $$\boxed{\int_0^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t = -\frac{\pi}{2}\ln 2.}$$

Exercise 6.7.3 *Compute*

$$g(x) = \int_0^{\pi} \ln(x^2 - 2x\cos t + 1)\,\mathrm{d}t$$

for $x \neq \pm 1$.

Correction 6.7.3

- *Continuity:*

  Let $f(x,t) = \ln(x^2 - 2x\cos t + 1)$ for $x \in \mathbb{R}\setminus\{-1,1\}$ and $t \in [0,\pi]$.
  By standard theorems, f is continuous on the product domain $D = \mathbb{R}\setminus\{-1,1\} \times [0,\pi]$ (noting that:

  $\forall x \in \mathbb{R}\setminus\{-1,1\},\ \forall t \in [0,\pi],\ x^2 - 2x\cos t + 1 > 0$).

  It follows that, if K is a segment contained in $\mathbb{R}\setminus\{-1,1\}$, we have

  $$\forall (x,t) \in K \times [0,\pi],\ |g(x,t)| \leq \|g\|_\infty^{K\times[0,\pi]} = \varphi(t)$$

  where the constant function φ is obviously integrable over $[0,\pi]$ (the infinity norm of g exists because it is continuous on a bounded closed set...).

  Conclusion: By the course theorem, g is continuous on $\mathbb{R}\setminus\{-1,1\}$.

- *Differentiability:*
$$\frac{\partial f}{\partial x}(x,t) = \frac{2x - 2\cos t}{x^2 - 2x\cos t + 1}$$
so $\dfrac{\partial f}{\partial x}(x,t)$ *is continuous on* D *(same arguments as above).*
By the course theorem (using the same method for domination), g *is* C^1 *on* $\mathbb{R}\setminus\{-1,1\}$, *and:*
$$\forall x > 0,\ g'(x) = \int_0^\pi \frac{2x - 2\cos t}{x^2 - 2x\cos t}\,\mathrm{d}t.$$

- *Calculation:*
First, let's calculate
$$J = \int_0^\pi \frac{\mathrm{d}t}{x^2 - 2x\cos t + 1}.$$
Setting $u = \tan\left(\frac{t}{2}\right)$ *(a* C^1 *diffeomorphism from* $[0,\pi[$ *onto* $[0,+\infty[$*), we obtain:*
$$\begin{aligned}
J &= \int_0^{+\infty} \frac{2\,\mathrm{d}u}{(1+u^2)(x^2 - 2x\left(\frac{1-u^2}{1+u^2}\right) + 1)} = \int_0^{+\infty} \frac{2\,\mathrm{d}u}{(1+u^2)(x^2+1) - 2x(1-u^2)} \\
&= \int_0^{+\infty} \frac{2\,\mathrm{d}u}{u^2(x+1)^2 + (x-1)^2} = \frac{1}{(x+1)^2}\int_0^{+\infty} \frac{2\,\mathrm{d}u}{u^2 + \left(\frac{x-1}{x+1}\right)^2} \\
&= \frac{2}{(x+1)^2}\left[\mathit{Arctan}\left(u\left|\frac{x+1}{x-1}\right|\right)\right]_0^{+\infty} = \frac{\pi}{|x^2-1|}.
\end{aligned}$$
We also have:
$$\begin{aligned}
K &= \int_0^\pi \frac{2\cos t}{x^2 - 2x\cos t + 1}\,\mathrm{d}t \\
&= \frac{1}{x}\int_0^\pi \frac{2x\cos t}{x^2 - 2x\cos t + 1}\,\mathrm{d}t \ \textit{(for } x \neq 0) \\
&= \frac{1}{x}\int_0^\pi \frac{2x\cos t - x^2 - 1 + x^2 + 1}{x^2 - 2x\cos t + 1}\,\mathrm{d}t \\
&= \frac{1}{x}\left(-\pi + (x^2+1)J\right).
\end{aligned}$$
Hence:
$$g'(x) = 2xJ - K = \frac{\pi}{x} + \left(2x - \frac{x^2+1}{x}\right)J = \frac{\pi}{x} + \pi\frac{x^2-1}{x}J = \frac{\pi}{x} + \pi\frac{x^2-1}{x|x^2-1|}.$$

- *Hence:*
  - *If* $x \in]-1,1[$, *then* $g'(x) = 0$, *so* $g(x) = \textit{constant} = g(0) = 0$.
  - *If* $|x| > 1$, *then* $g\left(\frac{1}{x}\right) = 0$ *(from the previous result, since* $\frac{1}{x} \in]-1,1[$*), so:*
$$0 = \int_0^\pi \ln\left(\frac{1}{x^2} - \frac{2}{x}\cos t + 1\right)\,\mathrm{d}t = \int_0^\pi \ln\left(\frac{1}{x^2}\left(1 - 2x\cos t + x^2\right)\right)\,\mathrm{d}t = \pi\ln\left(\frac{1}{x^2}\right) + g(x),$$
so finally:
$$g(x) = 2\pi\ln(|x|).$$

Exercise 6.7.4 *Calculate*
$$g(x) = \int_{-\infty}^{+\infty} e^{(-t^2+ixt)}\,\mathrm{d}t \quad \textit{for } x \in \mathbb{R}.$$

Correction 6.7.4
Existence and Continuity:
Let $f(x,t) = e^{-t^2+ixt}$ *for* $(x,t) \in \mathbb{R}^2$. *Clearly,* f *is continuous on* $\mathbb{R}^2$.
Moreover, for all $(x,t) \in \mathbb{R}^2$, $|f(x,t)| = e^{-t^2}$; *the function* $t \mapsto e^{-t^2}$ *is integrable on* $\mathbb{R}$, *so the domination hypothesis of the course theorem is satisfied, and* g *is continuous on* $\mathbb{R}$.

Derivative and Calculation:
For all $x \in \mathbb{R}$, *the function* $x \mapsto f(x,t)$ *is differentiable and:*

$$\frac{\partial f}{\partial x}(x,t) = ite^{-t^2+ixt},$$

which is a continuous function on $\mathbb{R}^2$. *Moreover,*

$$\left|\frac{\partial f}{\partial x}(x,t)\right| = te^{-t^2},$$

and $t \mapsto te^{-t^2}$ *is integrable on* $\mathbb{R}$, *so the domination hypothesis of the course theorem is satisfied, and* g *is of class* C^1 *on* $\mathbb{R}$.

Thus, we have:

$$\forall x \in \mathbb{R},\ g'(x) = \int_{-\infty}^{+\infty} ite^{-t^2+ixt}\,dt = \left[-\frac{i}{2}e^{-t^2+ixt}\right]_{-\infty}^{+\infty} - \frac{x}{2}\int_{-\infty}^{+\infty} e^{-t^2+ixt}\,dt$$

after integrating by parts (with $u = ie^{ixt}$ *and* $v' = te^{-t^2}$*). Since*

$$\lim_{t\to\pm\infty} e^{-t^2+ixt} = 0 \quad \textit{because} \quad \left|e^{-t^2+ixt}\right| = e^{-t^2},$$

we obtain

$$g'(x) = -\frac{x}{2}g(x),$$

which implies

$$g(x) = Ce^{-\frac{x^2}{4}},$$

where C *is a constant. Since* $g(0) = \sqrt{\pi}$ *(the Gaussian integral), we deduce that*

$$\boxed{\forall x \in \mathbb{R},\ g(x) = \sqrt{\pi}e^{-\frac{x^2}{4}}}.$$

Exercise 6.7.5 *Calculate:*

$$g(x) = \int_0^{+\infty} \frac{\arctan(xt)}{t(1+t^2)}\,dt$$

for $x \in \mathbb{R}$.

Correction 6.7.5
- <u>*Existence and Differentiability:*</u>

Let $f(x,t) = \dfrac{\arctan(xt)}{t(1+t^2)}$ *for* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. *We will directly apply the theorem from the course.*

a) *For a fixed* $x \in \mathbb{R}$, *the map* $t \mapsto f(x,t)$ *is continuous on* $\mathbb{R}^+$.

b) *For a fixed* $x \in \mathbb{R}$, *the map* $t \mapsto f(x,t)$ *is integrable on* $\mathbb{R}^+$ *because it is of constant sign, extendable by continuity at* 0^+ *(since* $\arctan u \sim u$ *as* $u \to 0$*), and asymptotically equivalent to* $\dfrac{const}{t^3}$ *as* $t \to +\infty$, *which is a Riemann-integrable function.*

c) For a fixed $t > 0$, the map $x \mapsto f(x,t)$ is of class C^1 on $\mathbb{R}$, and

$$\frac{\partial f}{\partial x}(x,t) = \frac{1}{(1+t^2)(1+x^2t^2)}.$$

d) We have the domination: $\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+$, $\left|\frac{\partial f}{\partial x}(x,t)\right| \leq \varphi(t)$, with $\varphi : t \mapsto \frac{1}{1+t^2}$, which is continuous and integrable on $\mathbb{R}^+$.

The theorem for differentiating an integral with a parameter gives that g is of class C^1 on $\mathbb{R}$, and

$$\forall x \in \mathbb{R},\ g'(x) = \int_0^{+\infty} \frac{dt}{(1+t^2)(1+x^2t^2)}.$$

- *Calculation:*

For $x \geq 0$ and $x \neq 1$, a partial fraction decomposition gives:

$$\frac{1}{(1+t^2)(1+x^2t^2)} = \frac{1}{1-x^2}\left(\frac{1}{1+t^2} - \frac{x^2}{1+x^2t^2}\right),$$

which leads to

$$g'(x) = \frac{\pi}{2(1+x)}.$$

This equality holds for $x = 1$ by continuity. Thus, we deduce that

$$g(x) = \frac{\pi}{2}\ln(1+x) \quad \textit{for} \quad x \in \mathbb{R}^+,$$

and by oddness, we have

$$g(x) = -\frac{\pi}{2}\ln(1-x) \quad \textit{for} \quad x \in \mathbb{R}^-.$$

Exercise 6.7.6 *Calculate:*

$$g(x) = \int_0^{+\infty} \frac{\ln(1+xt^2)}{1+t^2}\,\mathrm{d}t$$

pour $x \geqslant 0$

Correction 6.7.6

- *Existence:*

Let $f(x,t) = \dfrac{\ln(1+xt^2)}{1+t^2}$. For a fixed $x \geq 0$, the map $t \mapsto f(x,t)$ is continuous on $[0,+\infty[$. Moreover, we have $f(x,t) \sim \dfrac{2\ln t}{t^2}$ as $t \to +\infty$, so $f(x,t) = o\left(\frac{1}{t^{3/2}}\right)$ as $t \to +\infty$. Therefore, $t \mapsto f(x,t)$ is integrable on $\mathbb{R}^+$.

Thus, g is well-defined on $\mathbb{R}^+$.

- *Continuity:*

f is obviously continuous on $\mathbb{R}^+ \times \mathbb{R}^+$.
Let a be any strictly positive real number. Then, for $x \in [0,a]$,

$$0 \leq \frac{\ln(1+xt^2)}{1+t^2} \leq \frac{\ln(1+at^2)}{1+t^2} = \varphi(t),$$

where φ is continuous and integrable on $\mathbb{R}^+$.

The course theorem then implies the continuity of g on $[0,a]$. Since a is arbitrary, g is continuous on $\mathbb{R}^+$.

- *Differentiability:*

For all $x > 0$, the map $x \mapsto f(x,t)$ is differentiable, and

$$\frac{\partial f}{\partial x}(x,t) = \frac{t^2}{(1+xt^2)(1+t^2)},$$

which is a continuous function on $\mathbb{R}^+ \times \mathbb{R}^+$.

Let a be any strictly positive real number. Then, for $x \in [a, +\infty[$,

$$0 \leq \frac{\partial f}{\partial x}(x,t) \leq \frac{t^2}{(1+at^2)(1+t^2)} = \varphi(t),$$

where φ is continuous and integrable on $\mathbb{R}^+$ (since $\varphi(t) \sim \dfrac{1}{at^2}$ as $t \to +\infty$).

The course theorem then implies the differentiability of g on $[a, +\infty[$. Since a is arbitrary, g is differentiable on $\mathbb{R}^+$, and

$$\forall x > 0,\ g'(x) = \int_0^{+\infty} \frac{t^2}{(1+xt^2)(1+t^2)}\,dt.$$

A partial fraction decomposition gives, for $x > 0$ and $x \neq 1$:

$$g'(x) = \frac{1}{1-x}\int_0^{+\infty}\left(\frac{1}{1+xt^2} - \frac{1}{1+t^2}\right)dt = \frac{1}{1-x}\left(\left[\frac{1}{\sqrt{x}}\arctan(t\sqrt{x})\right]_{t=0}^{t=+\infty} - [\arctan(t)]_{t=0}^{t=+\infty}\right),$$

$$= \frac{\pi}{2}\frac{1}{\sqrt{x}(1+\sqrt{x})},$$

the equality holding for $x = 1$ by the continuity of g'.

Thus, for any real number $a > 0$,

$$g(x) = g(a) + \pi\int_a^x \frac{du}{2\sqrt{u}(1+\sqrt{u})} = g(a) + \pi\int_{\sqrt{a}}^{\sqrt{x}} \frac{dv}{1+v} = g(a) + \pi\left(\ln(1+\sqrt{x}) - \ln(1+\sqrt{a})\right).$$

This equality remains valid for $a = 0$ by the continuity of g; since $g(0) = 0$, we conclude:

$$\boxed{\forall x \geq 0,\ g(x) = \pi\ln(1+\sqrt{x}).}$$

Exercise 6.7.7 *Study of*

$$g : x \mapsto \int_0^{+\infty} \frac{e^{-xt}\sin t}{t}\,\mathrm{d}t$$

Correction 6.7.7

• *Study on $\mathbb{R}_+^*$:*

Let $f(x,t) = \dfrac{e^{-xt}\sin t}{t}\,dt$, for $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$. The function f is obviously continuous. Since $|\sin t| \leqslant |t|$, for any $x \in [a, +\infty[$ with $a > 0$ and any $t > 0$, we have:

$$|f(x,t)| \leqslant e^{-at} \quad \text{and} \quad \left|\frac{\partial f}{\partial x}(x,t)\right| = |e^{-xt}\sin t| \leqslant e^{-at},$$

where $t \mapsto e^{-at}$ is integrable on $\mathbb{R}^+$.

The hypotheses of the course theorem are thus satisfied on each interval $[a, +\infty[$ with $a > 0$, so g is of class C^1 on $\mathbb{R}^+$ and we have:

$$g'(x) = -\int_0^{+\infty} e^{-xt}\sin t\,dt = -Im\left(\int_0^{+\infty} e^{-xt+it}\,dt\right) = Im\frac{1}{i-x} = -\frac{1}{1+x^2}.$$

Thus: $g(x) = -Arctan(x) + constant$.
Moreover, since $|\sin t| \leqslant |t|$, *we have:*

$$|g(x)| \leqslant \int_0^{+\infty} e^{-xt}\,dt = \frac{1}{x}, \quad so \quad \lim_{x\to+\infty} g(x) = 0,$$

hence the constant is $\frac{\pi}{2}$.
Thus:

$$\boxed{\forall x > 0, \quad \int_0^{+\infty} \frac{e^{-xt}\sin t}{t}\,dt = \frac{\pi}{2} - Arctan(x)} \quad \left(= Arctan\frac{1}{x}\right).$$

- *Study at* 0^+*:*

It is slightly more difficult to prove that g *extends continuously to* $\mathbb{R}^+$, *by defining:*
$g(0) = \displaystyle\int_0^{+\infty} \frac{\sin t}{t}\,dt = \lim_{A\to+\infty} \int_0^A \frac{\sin t}{t}\,dt$ *(the existence of this limit was shown in class, and will be re-demonstrated below).*
For this, define:
$g(x) = g_1(x) + g_2(x)$, *where* $g_1(x) = \displaystyle\int_0^{\pi} \frac{e^{-xt}\sin t}{t}\,dt$ *and* $g_2(x) = \displaystyle\int_{\pi}^{+\infty} \frac{e^{-xt}\sin t}{t}\,dt$.

- Study of g_1:
  - *For all* $x \geqslant 0$, *the function* $t \mapsto f(x,t)$ *is continuous on* $]0,\pi]$ *(it is even extendable continuously at* 0*).*
  - *For all* $t \in]0,\pi]$, *the function* $x \mapsto f(x,t)$ *is continuous on* $\mathbb{R}^+$.
  - $\forall (x,t) \in \mathbb{R}^+ \times]0,\pi]$, $|f(x,t)| \leqslant 1$, *and the constant function equal to* 1 *is integrable on* $]0,\pi]$*!*

  By applying the course theorem, we deduce that g_1 *is continuous on* $\mathbb{R}^+$.

- Study of g_2:
  Integration by parts gives, for all $x \geqslant 0$, *knowing that a primitive of* $t \mapsto e^{-xt}\sin t$ *is the function* $t \mapsto -e^{-xt}\dfrac{\cos t + x\sin t}{(x^2+1)}$*:*

$$\int_{\pi}^{A} \frac{e^{-xt}\sin t}{t}\,dt = \left[-e^{-xt}\frac{\cos t + x\sin t}{t(x^2+1)}\right]_{\pi}^{A} - \frac{1}{x^2+1}\int_{\pi}^{A} e^{-xt}\frac{\cos t + x\sin t}{t^2}.$$

Now, $\displaystyle\lim_{t\to+\infty}\left(e^{-xt}\frac{\cos t + x\sin t}{t(x^2+1)}\right) = 0$ *(and this even for* $x = 0$*!) and, since the function* $t \mapsto e^{-xt}\dfrac{\cos t + x\sin t}{t^2}$ *is integrable on* $[\pi,+\infty[$ *(because it is bounded by* $\dfrac{constant}{t^2}$*), the limit as* $A \to +\infty$ *of the integral of this function on* $[\pi,+\infty[$ *exists. Finally, we have:*

$$\forall x \geqslant 0, \quad g_2(x) = -\frac{e^{-\pi x}}{\pi(x^2+1)} + \frac{1}{x^2+1} g_3(x),$$

where we have defined:

$$g_3(x) = \int_{\pi}^{+\infty} e^{-xt}\frac{\cos t + x\sin t}{t^2}\,dt.$$

Now, for all $x \in [0,B]$ *and all* $t \geqslant \pi$, *we have:*

$$\left|e^{-xt}\frac{\cos t + x\sin t}{t^2}\right| \leqslant \frac{x+1}{t^2} \leqslant \frac{B+1}{t^2},$$

and since $t \mapsto \dfrac{1}{t^2}$ *is integrable on* $[\pi,+\infty[$, *the course theorem guarantees the continuity of* g_3 *on* $\mathbb{R}^+$, *and hence* g_2 *is also continuous on* $\mathbb{R}^+$.

Conclusion: *g is continuous on* $\mathbb{R}^+$. *We deduce that:* $g(0) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \left(\frac{\pi}{2} - Arctan(x)\right)$, *so:*

$$\boxed{\int_0^{+\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}}.$$

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